BAND-LIMITED WAVELETS AND WAVELET PACKETS

by

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BAND-LIMITED WAVELETS AND WAVELET PACKETS

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CERTIFICATE

It is certified that the work contained in the thesis entitled, "BAND-LIMITED WAVELETS AND WAVELET PACKETS", by Biswaranjan Behera, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Modifications made in the thesis titled

"BAND-LIMITED WAVELETS AND WAVELET PACKETS"

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1. **Page 6, lines 5–6.** The sentence
   "In fact, such wavelets exist in plenty and are easy to construct".
   should be changed to the following:
   "Such wavelets exist in plenty and are easy to construct. In fact, the Shannon wavelet, presented in Example 1.2, gives an affirmative answer to the question".

2. **Page 8.** Corollary 1.2 should be rewritten as follows:
   Corollary 1.2. *Let K and W be subsets of $\mathbb{R}$, and W is both translation and dilation equivalent to K. Then W is a wavelet set if and only if K is so.*

3. **Page 10.** The last sentence of section 1.3 should be changed to:
   "In Chapter 2, we construct a large class of non-MRA wavelets; the two examples we just presented belong to this class".

4. **Page 16.** Remark 2.1.1 is incorrect and should be, therefore, deleted.

5. **Page 24.** The first sentence should be changed to the following:
   "We now present a proof of the last statement in Theorem 2.1".

6. At the end of Chapter 4, the following remark should be added.

   Remark 4.1. G. Majchrowska [Maj] has also constructed some new wavelet sets of $H^2(\mathbb{R})$. In particular, she proves the existence of an H^2-wavelet which does not belong to any $L^p(\mathbb{R})$, $p < 2$.

I am grateful to the thesis examiners for pointing out the reference [Maj] as well as for the suggestions for the above modifications in the thesis.

Additional Reference

[Maj] Majchrowska, G. (2001). *Some new examples of wavelets in the Hardy space $H^2(\mathbb{R})$*. Bulletin of the Polish Academy of Sciences: Mathematics, Vol. 49, No. 2.

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ERRATA

Page No.	Line	Change from	to
6	4	exists	exist
	-3	independendly	independently
15	-5	$-\frac{2}{3}$	$-\frac{2}{3}\pi$
16	1	$-\frac{2}{3}$	$-\frac{2}{3}\pi$
21	-4	$t_m(\xi)$	$t_q(\xi)$
27	2	p_{n+2}	p_{m+2}
	12	$[p_0, p_1]$	$[p_0, q_1]$
28	-3	$\left[\frac{2^{n-2}}{2^n-1}\pi, \frac{2^{n-2}-1}{2^n-1}\pi\right]$	$\left[\frac{2^{n-2}}{2^n-1}\pi, \frac{2^{n-1}-1}{2^n-1}\pi\right]$
	-3	$\left[\frac{2^{n-2}-1}{2^n-1}\pi, \frac{2^{n-2}}{2^n-1}\pi\right]$	$\left[\frac{2^{n-1}-1}{2^n-1}\pi, \frac{2^{n-1}}{2^n-1}\pi\right]$
33	-6	$[-d_n, -2^{n-1}]$	$[-d_n, -2^{n-1}\pi]$
35	2	$[-2\pi, \pi]$	$[-2\pi, -\pi]$
38	-7	$\frac{2^{n+2}}{3}\pi$	$-\frac{2^{n+2}}{3}\pi$
41	6	exist	exists
43	-5	$2^j\xi F_n$	$2^j\xi \in F_n$
45	2	θ on $[e_n, \pi]$	θ on $\mathbb{R}$
46	-9	$2^{j/2}(2^j \cdot -k)$	$2^{j/2}\psi(2^j \cdot -k)$
49	-6	k	k_1
59	1	$a^{j/2}\varphi_l(A \cdot -k)$	$a^{j/2}\varphi_l(A^j \cdot -k)$
	3	$a^{j/2}f(A \cdot -k)$	$a^{j/2}f(A^j \cdot -k)$
	3	$\hat{f}(B^{-1}\xi)$	$\hat{f}(B^{-j}\xi)$
	-4	a.e. ξ	a.e. $x \in \mathbb{R}^d$
61	4	H^r	H_r
67	7	$\hat{\Phi}_r$	$\hat{\Phi}$
68	-9	$F_n(B^{-1}\xi)$	$\hat{F}_n(B^{-1}\xi)$
	-8	$B^{(-j+1)}$	$B^{-(j+1)}$
74	-1 (1st term)	ψ^l	φ^l
	-1 (2nd term)	φ^l	ψ^l
77	3	$H_r r$	H_r
78	-7	maximal and minimal	minimal and maximal
81	10	$f(A^{-1}\cdot)$	$f(A^{-j}\cdot)$
82	-7	$k - A^jp$	$k + A^jp$
82	-6	$\|\|y - A^{-j}(k - A^jp)\|\|)\|\|$	$\|\|y - A^{-j}(k + A^jp)\|\|$

Note: Line -n means the n-th line from below.

Biswaranjan Behera

Synopsis

In this thesis we construct large classes of band-limited wavelets of $L^2(\mathbb{R})$ and the Hardy space $H^2(\mathbb{R})$. We have also considered the construction of multiwavelet packets and wavelet frame packets of $L^2(\mathbb{R}^d)$ with arbitrary dilation matrices.

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$, $d \geq 1$ is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-\imath\langle \xi, x\rangle}dx, \quad \xi \in \mathbb{R}^d.$$

To define the Fourier transform for functions of $L^2(\mathbb{R}^d)$, the operator $\mathcal{F}$ is extended from $L^1 \cap L^2(\mathbb{R}^d)$ to the whole of $L^2(\mathbb{R}^d)$. A function f is said to be *band-limited* if $\hat{f}$ is compactly supported.

Definition 0.1. *A function $\psi \in L^2(\mathbb{R})$ is said to be a wavelet of $L^2(\mathbb{R})$ if the system of functions $\{\ \psi_{j,k} : j, k \in \mathbb{Z}\ \}$ forms an orthonormal basis for $L^2(\mathbb{R})$, where*

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

G. Gripenberg and X. Wang independently proved the following theorem which characterizes all wavelets of $L^2(\mathbb{R})$.

Theorem 0.1. *A function ψ, with $\|\psi\|_2 = 1$, is a wavelet for $L^2(\mathbb{R})$ if and only if*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}.$$

$$\sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}\left(2^j(\xi + 2q\pi)\right)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and for all } q \in 2\mathbb{Z}+1.$$

Y. Meyer and S. Mallat defined the notion of a multiresolution analysis, which provides a framework to construct wavelets of $L^2(\mathbb{R})$. It is defined as follows.

Definition 0.2. *A multiresolution analysis (MRA) is a sequence of closed subspaces* $\{V_j : j \in \mathbb{Z}\}$ *of* $L^2(\mathbb{R})$ *satisfying the following properties:*

(i) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$

(ii) $\cup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$ *and* $\cap_{j\in\mathbb{Z}} V_j = \{0\}$

(iii) $f \in V_j$ *if and only if* $f(2\cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$

(iv) there exists a function $\varphi \in L^2(\mathbb{R})$ *such that* $\{\varphi(\cdot - k) : k \in \mathbb{Z}$ *forms an orthonormal basis for* V_0.

Given a wavelet ψ, define the subspaces $V_j = \overline{sp}\{\psi_{l,k} : l < j,\ k \in \mathbb{Z}\}$. If $\{V_j : j \in \mathbb{Z}\}$ is an MRA, then we say that ψ is an MRA-wavelet. Let

$$D_\psi(\xi) := \sum_{j\geq 1}\sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi + 2k\pi)\right)|^2, \quad \xi \in \mathbb{R} \tag{1}$$

The function D_ψ is called the dimension function of the wavelet ψ. The following characterization of MRA-wavelets was proved independently by Gripenberg and Wang.

Theorem 0.2. *A wavelet* ψ *is an MRA wavelet if and only if* $D_\psi = 1$ *a.e.*

Meyer constructed a wavelet such that both the wavelet and its Fourier transform are infinitely differentiable. The support of the Fourier transform of Meyer's wavelet is $[-\frac{8}{3}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \frac{8}{3}\pi]$. E. Hernández, X. Wang, and G. Weiss characterized all wavelets with Fourier transform supported in this set (see Theorem 4.1, Chapter 3 of [HW]). Our first result is the following. In Chapter 2, for $n \geq 1$, we construct a set S_n, which is symmetric with respect to the origin. The set S_1 is the Shannon set $[-2\pi, -\pi] \cup [\pi, 2\pi]$ whereas S_2 is the set $[-\frac{8}{3}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \frac{8}{3}\pi]$ associated with the Meyer wavelet. Then we characterize all wavelets with Fourier transform supported in S_n. Let $n \geq 1$. Put

$$a_n = \frac{2^{n-1}}{2^n - 1}\pi,\ b_n = \frac{2^n}{2^n - 1}\pi,\ c_n = \frac{2^{n-1}(2^n - 2)}{2^n - 1}\pi,\ d_n = \frac{2^{2n-1}}{2^n - 1}\pi, \text{ and } e_n = \frac{2^n - 2}{2^n - 1}\pi. \tag{2}$$

Define $S_n = S_n^+ \cup S_n^-$, where $S_n^+ = [a_n, b_n] \cup [c_n, d_n]$ and $S_n^- = -(S_n^+)$.

We prove the following theorem which characterizes all wavelets ψ such that $supp\ \hat{\psi} \subseteq S_n$.

Theorem 0.3. *Let* $n \geq 2$, $\psi \in L^2(\mathbb{R})$, *supp* $\hat{\psi} \subseteq S_n$ *and* $b(\xi) = |\hat{\psi}(\xi)|$. *Then* ψ *is a wavelet for* $L^2(\mathbb{R})$ *if and only if*

(i) $b(\xi) = 1$ *for a.e.* $\xi \in [a_n, e_n] \cup [-e_n, -a_n]$

(ii) $b^2(\xi) + b^2(2^{n-1}\xi) = 1$ *for a.e.* $\xi \in [e_n, b_n]$

(iii) $b^2(\xi) + b^2(\xi - 2\pi) = 1$ *for a.e.* $\xi \in [e_n, b_n]$

(iv) $b(\xi) = b\left(2^{n-1}(\xi - 2\pi)\right)$ *for a.e.* $\xi \in [e_n, b_n]$

(v) $\hat{\psi}(\xi) = e^{i\theta(\xi)}b(\xi)$, *where* θ *satisfies*

$$\theta(\xi) + \theta\left(2^{n-1}(\xi - 2\pi)\right) - \theta(\xi - 2\pi) - \theta(2^{n-1}\xi) = (2m(\xi) + 1)\,\pi,$$

for some $m(\xi) \in \mathbb{Z}$, *for a.e.* $\xi \in [e_n, b_n] \cap (supp\ b) \cap (\frac{1}{2^{n-1}} supp\ b)$.

Moreover, for $n \geq 3$, *none of these wavelets arises from an MRA.*

For $n \geq 3$, we have also computed the dimension functions D_n, defined in (1), of wavelets associated with the set S_n. These dimension functions are even and attain all integral values from 0 to the maximum value attained by D_n.

Chapter 3 deals with the constructions of wavelets with interesting properties. These wavelets are constructed with the help of the wavelets associated with the set S_n of Chapter 2. Our first result in this chapter is the construction of a family of band-limited wavelets whose Fourier transforms do not vanish in any neighbourhood of 0.

For $n \geq 2$, let a_n, b_n, c_n, d_n, and e_n be as in (2) and let $0 < \epsilon \leq \frac{a_n}{2}$. Define

$$P_1 = \left[\frac{a_n}{2} + \frac{\epsilon}{2^n}, \frac{a_n}{2} + \epsilon\right], \quad P_2 = [a_n + 2\epsilon, b_n], \quad \text{and} \quad P_3 = [d_n, d_n + 2\epsilon].$$

Let

$$X_0 = P_1 - 2 \cdot 2^{n-2}\pi, \quad Y_0 = \tfrac{1}{2^n}X_0,$$
$$X_l = Y_{l-1} - 2 \cdot 2^{n-2}\pi, \quad Y_l = \tfrac{1}{2^{n+l}}X_l, \quad l \geq 1.$$

Now define the set

$$W_n = \left(L \setminus \bigcup_{l=0}^{\infty} X_l\right) \cup \left(\bigcup_{l=0}^{\infty} Y_l\right) \cup M \cup (P_1 \cup P_2 \cup P_3). \tag{3}$$

Theorem 0.4. *For $n \geq 2$, the function ψ_n, defined by $\hat{\psi}_n = \chi_{W_n}$, is a band-limited wavelet such that $\hat{\psi}$ does not vanish in any neighbourhood of* 0.

In [Gar] G. Garrigós asked whether there exists a wavelet ψ such that (i) ψ is band-limited, i.e., $\hat{\psi}$ has compact support, (ii) $\hat{\psi}$ is even, and (iii) $\hat{\psi}$ does not vanish in any neighbourhood of the origin. Examples of wavelets satisfying any two of the above three properties exist in the literature. The next result is the construction of a family of wavelets satisfying all the three properties (i)-(iii). For $\epsilon > 0$ such that $0 < \epsilon \leq \frac{e_n}{4}$, construct the following sets:

$$S_1 = \left[\frac{a_n}{2} + \frac{\epsilon}{2^n}, \frac{a_n}{2} + \epsilon\right], \quad S_2 = [a_n + 2\epsilon, \pi], \quad S_3 = [d_n, d_n + 2\epsilon].$$

Let $T_i = -S_i$ for $i = 1, 2, 3$. Now, let

$$\begin{aligned} E_0 &= S_1 + 2 \cdot 2^{n-2}\pi, \quad F_0 = \tfrac{1}{2^{n+1}} E_0, \\ E_l &= F_{l-1} + 2 \cdot 2^{n-2}\pi, \quad F_l = \tfrac{1}{2^{n+l+1}} E_l, \quad l \geq 1, \\ G_l &= -E_l \text{ and } H_l = -F_l, \text{ for } l \geq 0. \end{aligned}$$

Define

$$\begin{aligned} K_n = \quad & \left(R_2 - \bigcup_{l=0}^{\infty} E_l\right) \cup \left(\bigcup_{l=0}^{\infty} F_l\right) \cup (S_1 \cup S_2 \cup S_3) \\ & \cup \left(L_2 - \bigcup_{l=0}^{\infty} G_l\right) \cup \left(\bigcup_{l=0}^{\infty} H_l\right) \cup (T_1 \cup T_2 \cup T_3). \end{aligned} \tag{4}$$

Theorem 0.5. *For each $n \geq 2$, the function w_n, defined by $\hat{w}_n = \chi_{K_n}$, is a wavelet which satisfies the above three properties.*

In [Web] E. Weber defined an equivalence relation on the set of all wavelets of $L^2(\mathbb{R})$ in the following manner. For a wavelet ψ, define the subspaces $V_j = \overline{sp}\{\psi_{j,k} : l < j, \ k \in \mathbb{Z}\}$. It is easy to verify that V_0 is invariant under translations by integers. Are there other

groups of translations under which V_0 is invariant? For $\alpha \in \mathbb{R}$, let T_α denote the unitary operator $T_\alpha f = f(\cdot - \alpha)$. Consider the group of translations $\mathcal{G}_n = \{T_{\frac{m}{2^n}} : m \in \mathbb{Z}\}$ and the group $\mathcal{G}_\infty = \{T_\alpha : \alpha \in \mathbb{R}\}$. Let $\mathcal{L}_n$ be the collection of all wavelets such that the corresponding space V_0 is invariant under the group $\mathcal{G}_n$. Then we have

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \mathcal{L}_n \supset \mathcal{L}_{n+1} \supset \cdots \supset \mathcal{L}_\infty.$$

An equivalence relation can now be defined on the collection of all wavelets of $L^2(\mathbb{R})$. The equivalence classes are given by $\mathcal{M}_n = \mathcal{L}_n \setminus \mathcal{L}_{n+1}$, with $\mathcal{M}_\infty = \mathcal{L}_\infty$. Therefore, $\mathcal{M}_n$ is the class of wavelets such that V_0 is invariant under the group $\mathcal{G}_n$ but not under $\mathcal{G}_{n+1}$. Weber characterized the classes $\mathcal{M}_n$ and constructed examples of wavelets in the classes $\mathcal{M}_\imath$, $\imath = 0, 1, 2, 3$. We have constructed examples of wavelets in each of the equivalence classes $\mathcal{M}_n$, $n \geq 0$.

Let $n \geq 3$ and a_n, b_n, c_n, d_n, and e_n be as in (2). Define ψ_n by

$$\hat{\psi}_n(\xi) = \begin{cases} 1 & \text{if } \xi \in [-b_n, -a_n] \cup [c_n, \frac{a_n}{2} + 2^{n-1}\pi] \cup [\frac{e_n}{2} + 2^{n-1}\pi, d_n] \\ \frac{1}{\sqrt{2}} & \text{if } \xi \in [\frac{a_n}{2}, \frac{e_n}{2}] \cup [a_n, e_n] \cup [\frac{a_n}{2} + 2^{n-1}\pi, \frac{e_n}{2} + 2^{n-1}\pi] \\ -\frac{1}{\sqrt{2}} & \text{if } \xi \in [a_n + 2^n\pi, e_n + 2^n\pi] \\ 0 & \text{otherwise.} \end{cases}$$

We prove the following theorem.

Theorem 0.6. *The function ψ_n, defined above, is a wavelet and belongs to the equivalence class $\mathcal{M}_{n-2}$.*

This theorem proves that $\mathcal{M}_n$, $n \geq 1$ are non-empty. We have also constructed a family of wavelets in the class $\mathcal{M}_0$.

Chapter 4 is on the construction of wavelets for the Hardy space $H^2(\mathbb{R})$ defined by

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } \xi < 0\}.$$

A function $\psi \in H^2(\mathbb{R})$ is said to be a wavelet of $H^2(\mathbb{R})$, or an H^2-wavelet, if the system of functions $\{\psi_{j,k} := 2^{j/2}(2^j \cdot -k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $H^2(\mathbb{R})$. The

function ψ such that $\hat{\psi} = \chi_{[2\pi,4\pi]}$ is one such example. In fact, this was the only known H^2-wavelet for a long time. A subset K of $\mathbb{R}_+$ is said to be an *H^2-wavelet set* if χ_K is the Fourier transform of an H^2-wavelet. An H^2-wavelet set will be called an *interval H^2-wavelet set* if it is the union of a finite number of intervals. In Chapter 4, we prove a result on the structure of interval H^2-wavelet sets and characterize all H^2-wavelet sets consisting of three disjoint intervals.

In Chapter 5, we extend the notion of wavelet packets, introduced by Coifman, Meyer and Wickerhauser, to $L^2(\mathbb{R}^d)$ with arbitrary dilation matrices and we have considered the case of the MRAs with multiplicity, i.e., for which the resolution spaces are spanned by more than one scaling function. The wavelet frame packets of [Che] is also generalized to $L^2(\mathbb{R}^d)$ with arbitrary dilations.

Let A be a *dilation matrix*, i.e., A is a $d \times d$ matrix such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, and all eigenvalues of A have absolute value greater than 1. For such a matrix A, $|\det A|$ is an integer greater than 1. Let $B = A^t$, the transpose of A and $a = |\det A| = |\det B|$. It is well known that the number of distinct cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is a. A subset K of $\mathbb{Z}^d$ which consists of exactly one element from each of these cosets will be called a *set of digits* for A.

Using the properties of MRA, we can find matrices $H_r(\xi)$, $0 \le r \le a-1$ of order L, having $2\pi\mathbb{Z}^d$-periodic functions as entries, such that the following relation holds:

$$\sum_{\mu\in K} H_r(\xi + 2B^{-1}\mu\pi)H_s^*(\xi + 2B^{-1}\mu\pi) = \delta_{rs}I_L \quad \text{for a.e. } \xi,\ 0 \le r,\ s \le a-1,$$

e I_L is the identity matrix of order L, $M^*(\xi)$ is the conjugate transpose of the matrix), and K is a set of digits for B. Let

$$H_r(\xi) = \left(h^r_{lj}(\xi)\right)_{1\le l,j\le L}, \qquad 0 \le r \le a-1,$$

$h^r_{lj}(\xi) = \sum_{k\in\mathbb{Z}^d} a^{-1/2} h^r_{ljk} e^{-\imath\langle\xi,k\rangle}$, where $h^r_{ljk} \in \mathbb{C}$. For $0 \le r \le a-1$, $1 \le l \le L$, define

$$f^r_l(x) = \sum_{j=1}^{L} \sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{1/2} \varphi_j(Ax - k).$$

Now, for any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define f_l^n, $1 \le l \le L$ recursively as follows. Suppose that f_l^r, $r \in \mathbb{N}_0$, $1 \le l \le L$ are defined already. Then define

$$f_l^{s+ar}(x) = \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{ljk}^s a^{1/2} f_j^r(Ax - k); \quad 0 \le s \le a-1, \ 1 \le l \le L. \tag{5}$$

Note that (5) defines f_l^n for every non-negative integer n and every l such that $1 \le l \le L$. The functions $\{f_l^n : n \ge 0, \ 1 \le l \le L\}$ will be called the *basic multiwavelet packets* corresponding to the MRA $\{V_j\}$. The first result regarding the wavelet packets is the following.

Theorem 0.7. *Let $\{f_l^n : n \ge 0, \ 1 \le l \le L\}$ be the basic multiwavelet packets. Then*

(i) $\{f_l^n(\cdot - k) : a^j \le n \le a^{j+1} - 1, \ 1 \le l \le L, \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of $W_j, j \ge 0$.

(ii) $\{f_l^n(\cdot - k) : 0 \le n \le a^j - 1, \ 1 \le l \le L, \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of $V_j, \ j \ge 0$.

(iii) $\{f_l^n(\cdot - k) : n \ge 0, \ 1 \le l \le L, \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Next, we have considered **all** dilations by the matrix A and **all** $\mathbb{Z}^d$-translations of the basic multiwavelet packet functions.

Definition 0.3. *Let $\{f_l^n : n \ge 0, \ 1 \le l \le L\}$ be the basic multiwavelet packets. The collection of functions $\mathcal{P} = \{a^{j/2} f_l^n(A^j \cdot -k) : n \ge 0, \ 1 \le l \le L, \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^d\}$ will be called the* general multiwavelet packets *associated with the MRA $\{V_j\}$ of multiplicity L.*

In the following theorem we characterize all subcollections of $\mathcal{P}$ which give rise to orthonormal bases of $L^2(\mathbb{R}^d)$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $I_{n,j} = \{r \in \mathbb{N}_0 : a^j n \le r \le a^j(n+1) - 1\}$.

Theorem 0.8. *Let $\{f_l^n : n \ge 0, \ 1 \le l \le L\}$ be the basic multiwavelet packets and $S \subset \mathbb{N}_0 \times \mathbb{Z}$. Then the collection of functions $\{a^{\frac{j}{2}} f_l^n(A^j \cdot -k) : 1 \le l \le L, k \in \mathbb{Z}^d, (n,j) \in S\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ if and only if $\{I_{n,j} : (n,j) \in S\}$ is a partition of $\mathbb{N}_0$.*

A sequence $\{x_k : k \in \mathbb{Z}\}$ of a separable Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist constants C_1 and C_2, $0 < C_1 \leq C_2 < \infty$ such that for all $x \in \mathcal{H}$

$$C_1\|x\|^2 \leq \sum_{k\in\mathbb{Z}} | \langle x, x_k \rangle |^2 \leq C_2\|x\|^2.$$

Similar to the orthogonal case, wavelet frame packets can be constructed. This was done for $L^2(\mathbb{R})$ in [Che] and for $L^2(\mathbb{R}^d)$ in [LC] for the dilation 2. We have extended these results to $L^2(\mathbb{R}^d)$ with arbitrary dilations and proved results similar to Theorem 0.7 and Theorem 0.8 in this setting.

Contents

Chapter 1

Introduction

1.1 A brief history

A function is said to be a wavelet if the collection of functions obtained by dyadic dilation and integral translations forms an orthonormal basis for $L^2(\mathbb{R})$. The first example of such a function was introduced by Haar in 1909. The disadvantage with this wavelet is that it is discontinuous. In 1927 Ph. Franklin constructed a system of continuous orthogonal functions which formed a basis of $L^2[0,1]$. J.O. Strömberg [Str] introduced the modified Franklin system in 1980. The Strömberg wavelet has the following properties: (i) it is continuous on the whole real line $\mathbb{R}$; (ii) it is linear on the intervals $[0, \frac{1}{2}], [\frac{1}{2}, 1]$ and on each interval of the form $[k, k+1], k \geq 1$ and of the form $[-\frac{l+1}{2}, -\frac{l}{2}], l \geq 0$; and (iii) ψ decreases rapidly at infinity. Morlet and Grossman used wavelets in the beginning of 1980's for their geological research. Y. Meyer [Mey1] constructed an example of a wavelet such that both the wavelet and its Fourier transform are infinitely differentiable. This was generalized to the higher dimensional spaces $L^2(\mathbb{R}^d)$ by Lemarié and Meyer in [LM1] In 1985 Meyer [Mey2], in collaboration with S. Mallat [Mal], introduced the concept of a multiresolution analysis which provided a framework to construct wavelets. Using this framework I. Daubechies in 1988 constructed, for each positive integer r, a wavelet which is compactly supported and r times continuously differentiable.

Let $\mathbb{T}$ denote the unit circle in the complex plane which can be identified with the interval $[-\pi, \pi)$. Let $L^2(\mathbb{T})$ be the space of all measurable, square-integrable functions defined on $\mathbb{T}$. It is a Hilbert space with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx.$$

The set of functions $\{e^{in\cdot} : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{T})$. By denoting $g(x) = e^{ix}$ and $g_n(x) = g(nx)$, we see that $L^2(\mathbb{T})$ is generated by the integral dilations of a single function g. That is, any function f in $L^2(\mathbb{T})$ can be represented as an infinite linear combination of the functions g_n:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f)g_n(x),$$

where the series on the right hand side converges in the norm of $L^2(\mathbb{T})$, which means that

$$\left\| f - \sum_{n=-N_1}^{N_2} c_n(f)g_n\right\|_2^2 := \frac{1}{2\pi}\int_0^{2\pi}\left|f(x) - \sum_{n=-N_1}^{N_2} c_n(f)g_n(x)\right|^2 dx \longrightarrow 0 \quad \text{as } N_1, N_2 \longrightarrow \infty.$$

The constants $c_n(f)$ are the Fourier coefficients of f defined by $c_n(f) = \langle f, g_n\rangle$.

If we want to analyse an arbitrary function f of $L^2(\mathbb{R})$, the space of square-integrable functions defined on the real line $\mathbb{R}$, with the help of functions generated by a single function, then such a function has to decay to 0 at $\pm\infty$ (obviously the functions g_n's cannot be used, as these functions themselves do not belong to $L^2(\mathbb{R})$). Therefore, we are looking for a small wave, or a *wavelet.* Since this wavelet will essentially be concentrated on a finite interval, to analyse an arbitrary function of $L^2(\mathbb{R})$ we have to shift the wavelet along the real line. That is, the translations of the wavelet have to be considered. Moreover, if a function oscillates too often (or, is spread out) in an interval which is very small (respectively, large) in comparison to the interval in which the wavelet is concentrated, then in order to analyse this function suitably, one has to contract (respectively, stretch) the wavelet upto a suitable order. That is, the wavelet has to be dilated. In other words, we are looking for a function ψ, which is smooth, localized and is such that the set of the following dilations and translations

$$\{\psi(2^j\cdot - k) : j, k \in \mathbb{Z}\}$$

suffices to analyse all functions of $L^2(\mathbb{R})$, as in the case of $L^2(\mathbb{T})$.

1.2 Basic concepts

In this section we list some of the definitions, examples and results which are well known in the theory of wavelets. We start with the precise definition of a wavelet.

Definition 1.1. *A function $\psi \in L^2(\mathbb{R})$ is said to be an orthonormal wavelet if the system of functions $\{ \psi_{j,k} : j, k \in \mathbb{Z} \}$ forms an orthonormal basis for $L^2(\mathbb{R})$, where*

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}. \tag{1.1}$$

In other words, the functions $\{\psi_{j,k}\}$ are orthonormal:

$$\langle \psi_{j,k}, \psi_{j',k'} \rangle = \delta_{j,j'}\delta_{k,k'},$$

and any function $f \in L^2(\mathbb{R})$ can be written as

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k},$$

where the convergence of the series is in the $L^2(\mathbb{R})$-norm. The inner product of two functions g and h of $L^2(\mathbb{R})$ is given by

$$\langle g, h \rangle = \int_{\mathbb{R}} g(x)\overline{h(x)}dx.$$

Since we will be concerned only with orthonormal wavelets, we will drop the word orthonormal in order to avoid repetitions. So, whenever we say that ψ is a wavelet, we mean that it is an orthonormal wavelet in the sense of Definition 1.1. The Fourier transform of a function plays a crucial role in the wavelet theory, which is defined for a function $f \in L^1(\mathbb{R}^d)$, $d \geq 1$ as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\langle \xi, x \rangle}dx, \quad \xi \in \mathbb{R}^d.$$

To define the Fourier transform for functions of $L^2(\mathbb{R}^d)$, the operator $\mathcal{F}$ is extended from $L^1 \cap L^2(\mathbb{R}^d)$, which is dense in $L^2(\mathbb{R}^d)$ in the L^2-norm, to the whole of $L^2(\mathbb{R}^d)$. For this definition of the Fourier transform, Plancherel theorem takes the form

$$\langle f, g\rangle = \frac{1}{(2\pi)^d}\langle \hat{f}, \hat{g}\rangle; \quad f, g \in L^2(\mathbb{R}^d).$$

Note that

$$(\psi_{j,k})^\wedge(\xi) = 2^{-\frac{j}{2}} e^{-i2^{-j}k\xi}\hat{\psi}(2^{-j}\xi), \quad \xi \in \mathbb{R},\ j, k \in \mathbb{Z}. \tag{1.2}$$

Two simple examples of wavelets are the Haar wavelet and the Shannon wavelet. The Haar wavelet is compactly supported in the time domain, i.e., in the x-variable and the Shannon wavelet is compactly supported in the frequency domain, i.e., in the ξ-variable.

Example 1.1. The Haar wavelet is defined by $\psi^H(x) = \chi_{[0,1/2]}(x) - \chi_{[1/2,1]}(x)$.

Example 1.2. The Shannon wavelet ψ^S. This wavelet is defined in terms of the Fourier transform:

$$\hat{\psi}^S = \chi_S, \quad S = [-2\pi, -\pi] \cup [\pi, 2\pi].$$

Observe that for each integer j,

$$(\psi^S_{j,k})^\wedge(\xi) = 2^{-\frac{j}{2}} e^{-i2^{-j}k\xi}\chi_{2^j S}(\xi), \quad k \in \mathbb{Z}.$$

Let the inner product in $L^2(S)$ be given by

$$\langle f, g\rangle = \int_S f(x)\overline{g(x)}dx.$$

Since $\left\{\frac{e^{-ik\cdot}}{\sqrt{2\pi}}\chi_S : k \in \mathbb{Z}\right\}$ is an orthonormal basis of $L^2(S)$, the collection of functions $\left\{\frac{2^{-j/2}}{\sqrt{2\pi}} e^{-i2^{-j}k\cdot}\chi_{2^j S} : k \in \mathbb{Z}\right\}$ is an orthonormal basis of $L^2(2^j S)$. Observe that $\{2^j S : j \in \mathbb{Z}\}$ is a pairwise disjoint collection of subsets of $\mathbb{R}$ whose union is $\mathbb{R}$. Therefore, by the above equality, $\{\frac{1}{\sqrt{2\pi}}(\psi^S_{j,k})^\wedge : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. By Plancherel theorem, $\{\psi^S_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. In other words, ψ^S is a wavelet. $\square$

A wavelet ψ, in particular, has the property that its integral translates form an orthonormal system. For such functions the following result holds.

Proposition 1.1. *Let $f \in L^2(\mathbb{R})$. Then $\{f(\cdot - k)\colon k \in \mathbb{Z}\}$ is an orthonormal system if and only if $\sum_{k\in\mathbb{Z}} |\hat{f}(\xi + 2k\pi)|^2 = 1$ for a.e $\xi \in \mathbb{R}$.*

Proof: Suppose that $\{f(\cdot - k)\colon k \in \mathbb{Z}\}$ is an orthonormal system. Then we have

$$\begin{aligned}
\delta_{k,0} &= \langle f, f(\cdot - k)\rangle \\
&= \frac{1}{2\pi}\left\langle \hat{f}, e^{-\imath k\cdot}\hat{f}\right\rangle \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 e^{\imath k\xi} d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big\{\sum_{l\in\mathbb{Z}} |\hat{f}(\xi + 2l\pi)|^2\Big\} e^{\imath k\xi} d\xi.
\end{aligned}$$

So the kth Fourier coefficient of the 2π-periodic function $\sum_{l\in\mathbb{Z}} |\hat{f}(\cdot + 2l\pi)|^2$ is $\delta_{k,0}$. Since the function which is 1 a.e. also has $\delta_{k,0}$ as its k-th Fourier coefficient, by uniqueness of the Fourier series it follows that $\sum_{l\in\mathbb{Z}} |\hat{f}(\xi + 2l\pi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$. By reversing the above steps we can prove the converse. $\square$

The support of a function g is defined as

$$supp\ g = \overline{\{x \in \mathbb{R} : g(x) \neq 0\}}.$$

The Lebesgue measure of a set E of $\mathbb{R}$ will be denoted by $|E|$. The following result gives information about the measure of the support of $\hat{f}$ (Corollary 2.4, Chapter 2 of [HW]).

Corollary 1.1. *Let $f \in L^2(\mathbb{R})$ and $\{f(\cdot - k)\colon k \in \mathbb{Z}\}$ is an orthonormal system. Then $|supp\ \hat{f}| \geq 2\pi$. Equality holds if and only if $|\hat{f}| = \chi_K$ for some measurable subset K of $\mathbb{R}$.*

Proof: From Proposition 1.1 it follows that $|\hat{f}(\xi)| \leq 1$ for a.e. $\xi \in \mathbb{R}$. Therefore,

$$2\pi = \|\hat{f}\|_2^2 = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi \leq \int_{(supp\ \hat{f})} 1 \cdot d\xi = |supp\ \hat{f}|,$$

which proves the first part. Now, suppose that $|supp\ \hat{f}| = 2\pi$. If $|\hat{f}(\xi)| < 1$ on a set E of positive measure, then

$$2\pi = \int_{(supp\ \hat{f})} |\hat{f}(\xi)|^2 d\xi < |supp\ \hat{f} \setminus E| + |E| = |supp\ \hat{f}| = 2\pi.$$

Hence, we get a contradiction which proves that $|\hat{f}(\xi)| = 1$ on *supp* $\hat{f}$. Conversely, suppose that $\{f(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system such that $|\hat{f}| = \chi_K$. Then

$$|K| = |supp\ \hat{f}| = ||\hat{f}||_2^2 = 2\pi.$$

□

A natural question to ask is the following: Does there exists a wavelet such that the support of its Fourier transform has the minimal measure 2π? In fact, such wavelets exist in plenty and are easy to construct. Such wavelets are called **minimally supported frequency** (MSF) wavelets. In [HKLS] these were called unimodular wavelets.

Definition 1.2. *A wavelet ψ is said to be an MSF wavelet if $|\hat{\psi}| = \chi_K$ for some measurable subset K of $\mathbb{R}$. The associated set K is said to be a wavelet set.*

A function $\psi \in L^2(\mathbb{R})$ is a wavelet of $L^2(\mathbb{R})$ if and only if ψ satisfies the following four conditions:

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{1.3}$$

$$\sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}\,(2^j(\xi + 2q\pi))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and for all } q \in 2\mathbb{Z}+1. \tag{1.4}$$

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{1.5}$$

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(\xi + 2k\pi)\overline{\hat{\psi}\,(2^j(\xi + 2k\pi))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and for all } j \geq 1. \tag{1.6}$$

The fact that equations (1.3)-(1.6) characterize all wavelets of $L^2(\mathbb{R})$ was observed by many authors. Lemarié and Meyer [LM2] obtained these equations for compactly supported wavelets. Bonami, Soria and Weiss [BSW] proved this fact under the assumption that ψ is band-limited, i.e., $\hat{\psi}$ is compactly supported. For a function ψ to be a wavelet, it is necessary that $||\psi||_2 = 1$. With this assumption on ψ, the following theorem was proved independently by G. Gripenberg and X. Wang (see also [HKLS]).

Theorem 1.1. *Let $\psi \in L^2(\mathbb{R})$ with $||\psi||_2 = 1$. Then ψ is a wavelet of $L^2(\mathbb{R})$ if and only if ψ satisfies* (1.3) *and* (1.4).

For a proof of this theorem we refer to Chapter 7 of [HW]. It was observed in [BSW] that the orthonormality of the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is characterized by (1.5) and (1.6), and completeness by (1.3) and (1.4) (see also [HW]). A consequence of Theorem 1.1 is that equations (1.3) and (1.4), along with the assumption $||\psi||_2 = 1$, imply the equations (1.5) and (1.6).

For MSF wavelets the characterization is simplified further, in the sense that only (1.3) and (1.5) are needed as observed by several authors (see [HKLS], [HW]).

Theorem 1.2. *Let $\psi \in L^2(\mathbb{R})$ be such that $|\hat{\psi}| = \chi_K$ for some measurable subset K of $\mathbb{R}$. Then ψ is a wavelet if and only if* (1.3) *and* (1.5) *hold.*

Proof: Since every wavelet must satisfy (1.3)-(1.6), we need only to show that if ψ is as in the hypothesis and satisfies (1.3) and (1.5), then it is a wavelet. We will show that (1.3) $\Rightarrow$(1.6) and (1.5) $\Rightarrow$(1.4). If $2^j\xi \in$ *supp* $\hat{\psi} = K$, then $|\hat{\psi}(2^j\xi)| = 1$. As $\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j\xi + 2k\pi)|^2 = 1$ by (1.5), we obtain that $\hat{\psi}(2^j\xi + 2k\pi) = 0$ for $k \neq 0$. In particular, $\hat{\psi}(2^j\xi + 2 \cdot 2^j q\pi) = 0$ for every odd integer q which proves that each term of the series in (1.4) is zero. Now if $\xi + 2k\pi \in K$, then $|\hat{\psi}(\xi + 2k\pi)| = 1$. Writing $\xi + 2k\pi$ instead of ξ in (1.3), we have $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 = 1$. So $\hat{\psi}(2^j(\xi + 2k\pi)) = 0$ for all $j \neq 0$. Hence, each term of the series in (1.6) is zero. $\square$

The above theorem for MSF wavelets can be restated in terms of the associated wavelet sets.

Theorem 1.3. *Let $K \subset \mathbb{R}$. Then K is a wavelet set if and only if*

1. *$\{K + 2k\pi : k \in \mathbb{Z}\}$ is an a.e. partition of $\mathbb{R}$.*

2. *$\{2^j K : j \in \mathbb{Z}\}$ is an a.e. partition of $\mathbb{R}$.*

We say that $\{A_l\}$ is an a.e. partition of X if $A_l \subset X$ for all l, $|A_l \cap A_{l'}| = 0$ for $l \neq l'$, and $|X \setminus \cup_l A_l| = 0$. To construct a wavelet set from another wavelet set, the concepts of dilation equivalence and translation equivalence of sets will be very helpful.

Definition 1.3. *A measurable set A is translation equivalent to a measurable set B if there exists a measurable partition $\{A_n\}$ of A and $k_n \in \mathbb{Z}$ such that $\{B_n\} \equiv \{A_n + 2k_n\pi\}$ is a partition of B. Similarly, a measurable set A is dilation equivalent to a measurable set B if there exists a measurable partition $\{A_n\}$ of A and $j_n \in \mathbb{Z}$ such that $\{B_n\} \equiv \{2^{j_n} A_n\}$ is a partition of B.*

As a consequence of Theorem 1.3 we have

Corollary 1.2. *Suppose K is a wavelet set and W is both translation and dilation equivalent to K. Then W is also a wavelet set.*

1.3 Multiresolution analysis

The concept of multiresolution analysis (MRA), formulated by Y. Meyer and S. Mallat, is crucial to the theory of wavelets. Given an MRA, one can always construct a wavelet of $L^2(\mathbb{R})$. An MRA is defined as follows.

Definition 1.4. *A multiresolution analysis (MRA) is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ satisfying the following properties:*

(i) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$

(ii) $\cup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$ *and* $\cap_{j\in\mathbb{Z}} V_j = \{0\}$

(iii) $f \in V_j$ *if and only if* $f(2\cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$

(iv) there exists a function $\varphi \in L^2(\mathbb{R})$ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis of V_0.

A function φ that satisfies property (iv) is called a *scaling function* for the MRA. If φ_1 is another scaling function for the same MRA, then $\hat{\varphi}_1 = l \cdot \hat{\varphi}$, where l is a 2π-periodic function in $L^2(\mathbb{T})$ such that $|l(\xi)| = 1$ a.e. (see Lemma 2.6, Chapter 2 of [HW]). Note that $\{2^{\frac{j}{2}}\varphi(2^j \cdot -k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_j because of (iii) and (iv).

We will now briefly mention how to get a wavelet from an MRA. Since $\varphi \in V_0 \subset V_1$ and $\{2^{\frac{1}{2}}\varphi(2\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_1, there exists $\{c_k : k \in \mathbb{Z}\} \in l^2(\mathbb{Z})$ such that

$$\varphi(x) = \sum_{k\in\mathbb{Z}} c_k 2^{\frac{1}{2}}\varphi(2x - k).$$

Taking Fourier transforms of both sides, we get

$$\hat{\varphi}(\xi) = \sum_{k\in\mathbb{Z}} c_k 2^{-\frac{1}{2}} e^{-ik\frac{\xi}{2}} \hat{\varphi}\left(\frac{\xi}{2}\right) = m_0\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right), \tag{1.7}$$

where

$$m_0(\xi) = 2^{-\frac{1}{2}} \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}.$$

The function m_0, which is 2π-periodic and is in $L^2(\mathbb{T})$, is called the *low pass filter* associated with the scaling function φ. By Proposition 1.1 we have $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$. Splitting the sum over $\mathbb{Z}$ into sums over even and odd integers and making use of (1.7), we obtain

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1 \quad \text{for a.e } \xi \in \mathbb{T}. \tag{1.8}$$

Let W_j be the orthogonal complement of V_j in V_{j+1}:

$$W_j = V_{j+1} \ominus V_j, \quad j \in \mathbb{Z}.$$

By properties *(i)* and *(ii)* of Definition 1.4 we can decompose $L^2(\mathbb{R})$ as an orthogonal direct sum of the W_j's:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j. \tag{1.9}$$

It follows from *(iii)* that if $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_0, then $\{2^{\frac{j}{2}}\psi(2^j \cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_j. Then by the decomposition (1.9) it follows that ψ is a wavelet. Therefore, all that is needed is to find a function $\psi \in W_0$ such that $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_0. Functions ψ with this property are characterized by

$$\hat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\nu(\xi)\overline{m_0\left(\frac{\xi}{2} + \pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right), \quad \text{for a.e } \xi \in \mathbb{R}, \tag{1.10}$$

where ν is a 2π-periodic measurable function such that $|\nu(\xi)| = 1$ for a.e. $\xi \in \mathbb{T}$, φ is the scaling function of the MRA and m_0 is the low pass filter associated with φ (see Proposition 2.13, Chapter 2 of [HW] for details).

Given a wavelet ψ, define the spaces $V_j = \overline{sp}\{\psi_{l,k} : l < j, k \in \mathbb{Z}\}$. If $\{V_j : j \in \mathbb{Z}\}$ is an MRA, then we say that ψ is associated with an MRA (or, ψ is an MRA-wavelet). Not all wavelets are associated with an MRA. The first example of a non-MRA wavelet, which is an MSF wavelet, was given by J.L. Journé. The associated wavelet set is

$$J = \left[-\frac{32}{7}\pi, -4\pi\right] \cup \left[-\pi, -\frac{4}{7}\pi\right] \cup \left[\frac{4}{7}\pi, \pi\right] \cup \left[4\pi, \frac{32}{7}\pi\right].$$

Another interesting example is the Lemarié wavelet which is also an MSF wavelet with the following wavelet set:

$$L = \left[-\frac{8}{7}\pi, -\frac{4}{7}\pi\right] \cup \left[\frac{4}{7}\pi, \frac{6}{7}\pi\right] \cup \left[\frac{24}{7}\pi, \frac{32}{7}\pi\right].$$

In Chapter 2 we will show that these two wavelets cannot be constructed from any MRA.

1.4 The dimension function

There is a single equation which characterizes all wavelets associated with MRAs. Before stating this result let us see how the Fourier transform $\hat{\varphi}$ of the scaling function of an MRA is related with the Fourier transform $\hat{\psi}$ of the corresponding wavelet. From (1.7), (1.8) and (1.10) we get

$$|\hat{\varphi}(2\xi)|^2 + |\hat{\psi}(2\xi)|^2 = \left(|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2\right) \cdot |\hat{\varphi}(\xi)|^2 = |\hat{\varphi}(\xi)|^2.$$

Iterating this equation, we obtain

$$|\hat{\varphi}(\xi)|^2 = |\hat{\varphi}(2^N\xi)|^2 + \sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2 \quad \text{for all } N \geq 1.$$

Since $|\hat{\varphi}(\xi)| \leq 1$, the sequence $\left\{\sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2 : N \geq 1\right\}$, which is an increasing sequence of real numbers, is bounded above by 1 so that $\lim_{N\to\infty} \sum_{j=1}^{N} |\hat{\psi}(2^j\xi)|^2$ exists. Therefore,

$\lim_{N\to\infty} |\hat{\varphi}(2^N\xi)|^2$ also exists. Now by Fatou's lemma, we have

$$\begin{aligned}\int_{\mathbb{R}} \lim_{N\to\infty} |\hat{\varphi}(2^N\xi)|^2 d\xi &\le \lim_{N\to\infty} \int_{\mathbb{R}} |\hat{\varphi}(2^N\xi)|^2 d\xi \\ &= \lim_{N\to\infty} \frac{1}{2^N} \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 d\xi \\ &= \lim_{N\to\infty} \frac{1}{2^N} \cdot 2\pi \\ &= 0.\end{aligned}$$

Hence,

$$|\hat{\varphi}(\xi)|^2 = \sum_{j\ge 1} |\hat{\psi}(2^j\xi)|^2 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{1.11}$$

Using Proposition 1.1, we obtain

$$D_\psi(\xi) \equiv \sum_{j\ge 1}\sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi + 2k\pi)\right)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{1.12}$$

Therefore, for a wavelet ψ to be associated with an MRA, it is necessary that the function D_ψ is equal to 1 a.e. The fact that it is also sufficient is proved independently by Gripenberg and Wang. We refer to Chapter 7 of [HW] for a proof of the following theorem.

Theorem 1.4. *A wavelet ψ is an MRA wavelet if and only if $D_\psi = 1$ for a.e. $\xi \in \mathbb{R}$.*

The function D_ψ is called the *dimension function* of ψ, since for a.e. ξ, $D_\psi(\xi)$ happens to be the dimension of certain subspaces of $l^2(\mathbb{Z})$ (see [Aus]). This fact, in particular, shows that $D_\psi(\xi)$ is an integer for a.e. ξ. The significance of the dimension function is that, given a dimension function one can construct wavelets associated with it. In [BRS], a characterization of dimension functions is given. An algorithm to construct wavelets from the dimension function is also provided therein.

1.5 Wavelets for $H^2(\mathbb{R})$

The space $H^2(\mathbb{R})$ is the collection of all functions in $L^2(\mathbb{R})$ whose Fourier transform vanish on the negative real axis:

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \quad \text{for a.e. } \xi \le 0\}.$$

It is clear that $H^2(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$. As in the case of $L^2(\mathbb{R})$, we can define wavelets for $H^2(\mathbb{R})$. *A function $\psi \in H^2(\mathbb{R})$ is said to be a wavelet for $H^2(\mathbb{R})$ (or, an H^2-wavelet) if the system of functions $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $H^2(\mathbb{R})$.*

Example 1.3. The function ψ given by $\hat{\psi} = \chi_{[2\pi,4\pi]}$ is an H^2-wavelet. The proof of this fact is easy and is, in fact, similar to the case of Shannon wavelet of $L^2(\mathbb{R})$ (see Example 1.2).

Analogous to the $L^2(\mathbb{R})$ case, we call a set $K \subset (0, \infty)$ an H^2-wavelet set if χ_K is the Fourier transform of an H^2-wavelet. If K is a union of a finite number of intervals then K will be called an interval wavelet set. In [HKLS] the authors characterized all H^2-wavelets ψ such that $|\hat{\psi}|$ is the characteristic function of either an interval or is the union of two disjoint intervals. In Chapter 4 we will prove a result on the structure of the interval wavelet sets of $H^2(\mathbb{R})$ and characterize all such sets which are unions of three disjoint intervals.

1.6 Multiwavelet packets of $L^2(\mathbb{R}^d)$

Let us recall that a function ψ is a wavelet of $L^2(\mathbb{R})$ if $\{2^{j/2}\psi(2^j \cdot -k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. In order to extend this concept to higher dimensional spaces $L^2(\mathbb{R}^d)$, $d > 1$, the notions of dilation and translation have to be generalized. The natural generalization is dilation by powers of 2 in every coordinate, i.e., $x = (x_1, x_2, \ldots, x_d) \mapsto 2^j x = (2^j x_1, 2^j x_2, \ldots, 2^j x_d)$, and translation by elements of $\mathbb{Z}^d$. Then one can define MRA for $L^2(\mathbb{R}^d)$. But we can do more: we can replace the dyadic dilation by a general dilation matrix (i.e., a $d \times d$ matrix which leaves $\mathbb{Z}^d$ invariant, and such that all its eigenvalues have absolute value greater than 1). Moreover, in the definition of MRA, we can allow the central space V_0 to be generated by more than one scaling function. Such an MRA is called an MRA with multiplicity and the wavelets associated with the MRA are called multiwavelets.

Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R})$ and $W_j = V_{j+1} \ominus V_j$ be the corresponding wavelet subspaces so that the orthogonal decomposition $L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j$ holds. If ψ is a wavelet associated with this MRA, then $\{\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_j and consequently, $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. If ψ is band-limited, i.e., $\hat{\psi}$ has compact support, then $|supp\ \hat{\psi}_{j,k}| = 2^j|supp\ \hat{\psi}|$ which shows that the wavelets have poor frequency localizations (the supports of $\hat{\psi}_{j,k}$ gets larger for large j). In order to overcome this disadvantage, Coifman, Meyer, and Wickerhauser constructed wavelet packets from an MRA wavelet basis. The idea is to decompose the wavelet subspaces W_j into 2^j orthogonal components so that each component is generated by only the integer translates of a function. In this manner one can get better frequency localization. The notion of wavelet packets can be extended to higher dimensions.

1.7 Organization of the thesis

This thesis can be divided into two parts. The first part (which consists of Chapters 2-4) is on the constructions of band-limited wavelets of $L^2(\mathbb{R})$ and the Hardy space $H^2(\mathbb{R})$. The second part (Chapter 5) is on multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$ with arbitrary dilations.

In Chapter 2 we construct a set S_n for each $n \geq 1$ and then we characterize all wavelets with Fourier transform supported in S_n. The case $n = 1$ corresponds to the Shannon wavelet whereas S_2 is the set associated with the Lemarié- Meyer wavelets. Moreover, we show that, if $n \geq 3$, then none of the wavelets associated with S_n is an MRA-wavelet. The case $n = 3$ also includes the Journé wavelet and the Lemarié wavelet presented in §1.3. We have also computed the dimension functions D_n associated with these wavelets. The functions D_n turn out to be even functions and they assume all values from 0 to the maximum value attained by D_n. In Chapter 3 we construct, with the help of the wavelets characterized in Chapter 2, wavelets satisfying interesting properties. One of these families answers in positive a question posed in [Gar] about the existence

of wavelets with certain properties. We also show that each of the equivalence classes $\mathcal{M}_n, n \geq 0$ of wavelets defined in [Web] is non-empty. In addition, a family of wavelets in $\mathcal{M}_0$ is constructed. Chapter 4 deals with the construction of H^2-wavelets. We prove a result on the structure of interval wavelet sets of $H^2(\mathbb{R})$ and characterize all wavelet sets of $H^2(\mathbb{R})$ which are unions of exactly three intervals. Finally, in Chapter 5 we study the multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$ where we have taken arbitrary dilation matrices and allowed the resolution spaces to be spanned by several scaling functions.

Chapter 2

A class of band-limited wavelets of $L^2(\mathbb{R})$

In this chapter we will construct a class of band-limited wavelets of $L^2(\mathbb{R})$. For each integer $n \geq 1$ we construct a set S_n and characterize all wavelets whose Fourier transform is supported in S_n. Then we show that these wavelets are not associated with any MRA.

2.1 Introduction

A function is said to be band-limited if its Fourier transform has compact support. The simplest example of a band-limited wavelet is the Shannon wavelet whose Fourier transform is the characteristic function of the set $S = [-2\pi, -\pi] \cup [\pi, 2\pi]$. Lemarié and Meyer ([Mey1], [LM1]) constructed a wavelet whose Fourier transform is infinitely differentiable. This wavelet satisfies $\hat{\psi}(\xi) = e^{i\xi/2}b(\xi)$, where b is an even, non-negative "bell-shaped" function and its support is equal to the set $[-\frac{8}{3}\pi, -\frac{2}{3}] \cup [\frac{2}{3}\pi, \frac{8}{3}\pi]$. In [BSW] the authors characterized all wavelets ψ such that $|\hat{\psi}|$ is an even continuous function whose support is equal to the set $[-2\pi-\epsilon', -\pi+\epsilon] \cup [\pi-\epsilon, 2\pi+\epsilon']$, where $\epsilon, \epsilon' > 0$ and $\epsilon+\epsilon' \leq \pi$, so that this class includes the Lemarié-Meyer wavelet as a particular case. Later, in [HWW1] (see also Theorem 4.1, Chapter 3 of [HW]), the restriction on ψ (i.e., $|\hat{\psi}|$ is even and continuous)

was removed and all wavelets with Fourier transform supported in $[-\frac{8}{3}\pi, -\frac{2}{3}] \cup [\frac{2}{3}\pi, \frac{8}{3}\pi]$ were characterized[1].

A special case of band-limited wavelets, the MSF wavelets, were considered in [HKLS] in which these were called the unimodular wavelets. Recall that for an MSF wavelet ψ, there exists a measurable set K of $\mathbb{R}$ such that $|\hat{\psi}| = \chi_K$. Denote $K^+ = K \cap \mathbb{R}^+$ and $K^- = K \cap \mathbb{R}^-$. The authors of [HKLS] characterized all MSF wavelets for which the associated wavelet set K is symmetric with respect to 0 and K^+ is a union of two disjoint intervals. They proved that these are precisely those wavelets for which $K^+ = \left[\frac{2^{n-1}}{2^n-1}\pi, \pi\right] \cup \left[2^n\pi, \frac{2^{2n-1}}{2^n-1}\pi\right]$ for some $n \geq 1$. This includes the Journé wavelet.

This chapter is organized as follows. In §2.2 we will construct a set S_n for each $n \geq 1$, which is symmetric with respect to the origin and is a union of four intervals. In §2.3 all wavelets ψ with Fourier transform supported in S_n will be characterized. Next, in §2.4 we show that none of these wavelets is associated with an MRA. For a wavelet to be associated with an MRA, it is necessary and sufficient that the corresponding dimension function is equal to 1 a.e. In §2.5 we will compute the dimension function explicitly and show that it is an even function and takes all integral values from 0 to the maximum value it attains. This will provide an alternative proof of the fact that the wavelets are not MRA-wavelets.

2.2 The set S_n

Let $n \geq 1$. Put

$$a_n = \frac{2^{n-1}}{2^n-1}\pi, \qquad b_n = 2a_n = \frac{2^n}{2^n-1}\pi,$$
$$c_n = \frac{2^{n-1}(2^n-2)}{2^n-1}\pi, \quad d_n = 2^n a_n = \frac{2^{2n-1}}{2^n-1}\pi.$$

Define $S_n = S_n^+ \cup S_n^-$, where $S_n^+ = [a_n, b_n] \cup [c_n, d_n]$ and $S_n^- = -(S_n^+)$.

Remark 2.1. 1. If $n = 1$, then $a_1 = \pi$, $b_1 = c_1 = d_1 = 2\pi$ and S_1 is the support of the Fourier transform of the Shannon wavelet.

[1] In fact, they characterized all wavelets with Fourier transform supported in the set $[-\frac{8}{3}a, 4\pi - \frac{4}{3}a]$, $0 < a \leq \pi$. It was also shown that for such a wavelet ψ, it is necessary that $\hat{\psi} = 0$ on $[-\frac{2}{3}a, 2\pi - \frac{4}{3}a]$.

2. For $n = 2$, $a_2 = \frac{2}{3}\pi$, $b_2 = c_2 = \frac{4}{3}\pi$ and $d_2 = \frac{8}{3}\pi$. So S_2 is the set associated with the Lemarié-Meyer wavelets.

3. As n increases, the set $[a_n, b_n]$ moves closer to 0 whereas $[c_n, d_n]$ moves farther away. Also observe that the measure of the set S_n is $2(\frac{2^{n-1}}{2^n-1}\pi + \frac{2^n}{2^n-1}\pi)$ which approaches 3π as n tends to infinity.

All terms in the equations (1.3)–(1.6) that characterize all wavelets of $L^2(\mathbb{R})$ involve dilations by powers of 2 and translations by integral multiples of 2π. In the following lemma, for various subsets of S_n, we identify those translates and dilates which are possibly in S_n. We present it in a tabular form for easy reference.

Lemma 2.1. *For $n \geq 3$, let S_n be as above, and let $e_n = \frac{2^n-2}{2^n-1}\pi$. Then*

$\xi \in$ *the set*	$\xi + 2k\pi \notin S_n$ *unless*	$2^j\xi \notin S_n$ *unless*
$[a_n, e_n]$	$k = 0$	$j = 0$
$[e_n, b_n]$	$k = 0$ or -1	$j = 0$ or $n-1$
$[c_n, d_n]$	$k = 0$ or -2^{n-1}	$j = 0$ or $-(n-1)$
$[-e_n, -a_n]$	$k = 0$	$j = 0$
$[-b_n, -e_n]$	$k = 0$ or 1	$j = 0$ or $n-1$
$[-d_n, -c_n]$	$k = 0$ or 2^{n-1}	$j = 0$ or $-(n-1)$

Also observe that

$$[e_n, b_n] - 2\pi = [-b_n, -e_n], \tag{2.1}$$

$$2^{n-1}[e_n, b_n] = [c_n, d_n],$$

and

$$[c_n, d_n] - 2 \cdot 2^{n-1}\pi = [-d_n, -c_n].$$

2.3 The characterization

The following theorem characterizes all wavelets ψ such that *supp* $\hat{\psi}$ is contained in S_n. The case $n = 2$ is Theorem 4.1 of Chapter 3 in [HW] in which case the wavelets are MRA wavelets.

Theorem 2.1. *Let $n \geq 2$, $\psi \in L^2(\mathbb{R})$, supp $\hat{\psi} \subseteq S_n$ and $b(\xi) = |\hat{\psi}(\xi)|$. Then ψ is a wavelet for $L^2(\mathbb{R})$ if and only if*

(i) $b(\xi) = 1$ *for a.e.* $\xi \in [a_n, e_n] \cup [-e_n, -a_n]$

(ii) $b^2(\xi) + b^2(2^{n-1}\xi) = 1$ *for a.e.* $\xi \in [e_n, b_n]$

(iii) $b^2(\xi) + b^2(\xi - 2\pi) = 1$ *for a.e.* $\xi \in [e_n, b_n]$

(iv) $b(\xi) = b\left(2^{n-1}(\xi - 2\pi)\right)$ *for a.e.* $\xi \in [e_n, b_n]$

(v) $\hat{\psi}(\xi) = e^{i\theta(\xi)}b(\xi)$, *where θ satisfies*

$$\theta(\xi) + \theta\left(2^{n-1}(\xi - 2\pi)\right) - \theta(\xi - 2\pi) - \theta(2^{n-1}\xi) = (2m(\xi) + 1)\,\pi,$$

for some $m(\xi) \in \mathbb{Z}$, for a.e. $\xi \in [e_n, b_n] \cap (supp\ b) \cap (\frac{1}{2^{n-1}} supp\ b)$.

Moreover, for $n \geq 3$, none of these wavelets arises from an MRA.

Remark 2.2. 1. Observe that it follows from the above theorem that $|\hat{\psi}|$ is completely determined by its values on $[e_n, b_n]$. On this set let $b = |\hat{\psi}|$ be an arbitrary function taking values between 0 and 1. Using the relations following Lemma 2.1, $|\hat{\psi}|$ can be extented to other sets of S_n with the help of properties $(i) - (iv)$. Properties (ii), (iii) and (iv) can also be written as

$$\begin{aligned} b^2(\xi) + b^2\Big(\frac{1}{2^{n-1}}\xi\Big) &= 1 \quad \text{a.e. on } [c_n, d_n] && (2.2)\\ b^2(\xi) + b^2(\xi + 2\pi) &= 1 \quad \text{a.e. on } [-b_n, -e_n] && (2.3)\\ b\left(\frac{1}{2^{n-1}}\xi + 2\pi\right) &= b(\xi) \quad \text{a.e. on } [-d_n, -c_n]. && (2.4) \end{aligned}$$

2. If $(supp\ b) \cap (\frac{1}{2^{n-1}} supp\ b)$ has an empty interior in $[e_n, b_n]$, then θ can be chosen to be any measurable function. In particular, we can take $\theta(\xi) = 0$.

To prove Theorem 2.1 we will use Theorem 1.1 which characterizes all wavelets of $L^2(\mathbb{R})$.

Proof of Theorem 2.1

First let us assume that ψ is a wavelet for $L^2(\mathbb{R})$, $supp\ \hat{\psi} \subseteq S_n$ and $b = |\hat{\psi}|$. So ψ satisfies equations (1.3)–(1.6). Consider equality (1.5) :

$$\sum_{k\in\mathbb{Z}} b^2(\xi + 2k\pi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}.$$

We use Lemma 2.1 to pick out the non-zero terms in this sum. If $|\xi| \in [a_n, e_n]$, then only $k = 0$ will contribute to the sum. So we get $b(\xi) = 1$ a.e., which is (i) of the theorem. For $\xi \in [e_n, b_n]$, we get non-zero contributions from $k = 0$ and $k = -1$. That is, $b^2(\xi) + b^2(\xi - 2\pi) = 1$ for a.e. ξ which proves (iii). Also if $\xi \in [e_n, b_n]$, then $2^{n-1}\xi \in [c_n, d_n]$, and we get (non-zero terms now correspond to $k = 0$ and $k = -2^{n-1}$)

$$b^2(2^{n-1}\xi) + b^2(2^{n-1}\xi - 2^n\pi) = 1 \quad \text{for a.e. } \xi \in [e_n, b_n]. \tag{2.5}$$

This can also be written as

$$b^2(\xi) + b^2(\xi - 2^n\pi) = 1 \quad \text{for a.e. } \xi \in [c_n, d_n]. \tag{2.6}$$

Now consider equality (1.3):

$$\sum_{j\in\mathbb{Z}} b^2(2^j\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}.$$

For $\xi \in [e_n, b_n]$, the only possible non-zero terms correspond to $j = 0$ and $j = n - 1$. So we conclude $b^2(\xi) + b^2(2^{n-1}\xi) = 1$. This proves (ii). Combining (ii) and (2.5) we get condition (iv). It remains to prove condition (v). For this purpose we consider equality (1.6) with $j = n - 1$:

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(\xi + 2k\pi)\overline{\hat{\psi}\,(2^{n-1}(\xi + 2k\pi))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}.$$

If $\xi \in [e_n, b_n]$, the non-zero terms in the sum come from $k = 0$ and -1. Hence,

$$\hat{\psi}(\xi)\overline{\hat{\psi}(2^{n-1}\xi)} + \hat{\psi}(\xi - 2\pi)\overline{\hat{\psi}\left(2^{n-1}(\xi - 2\pi)\right)} = 0 \quad \text{for a.e. } \xi \in [e_n, b_n]. \tag{2.7}$$

Using (iii), (2.5) and (2.7), we see that for almost every $\xi \in [e_n, b_n]$, the vectors

$$\left(\hat{\psi}(\xi), \hat{\psi}(\xi - 2\pi)\right) \quad \text{and} \quad \left(\hat{\psi}(2^{n-1}\xi), \hat{\psi}(2^{n-1}(\xi - 2\pi))\right)$$

are orthonormal in $\mathbb{C}^2$. If we let $\hat{\psi}(\xi) = e^{i\theta(\xi)}b(\xi)$, then it follows that, for some real-valued measurable function α,

$$\begin{aligned} &e^{i\alpha(\xi)}\left(e^{i\theta(\xi)}b(\xi), e^{i\theta(\xi-2\pi)}b(\xi - 2\pi)\right) \\ &\quad = \left(-e^{-i\theta(2^{n-1}(\xi-2\pi))}b(2^{n-1}(\xi - 2\pi)), e^{-i\theta(2^{n-1}\xi)}b(2^{n-1}\xi)\right), \end{aligned} \tag{2.8}$$

for a.e. $\xi \in [e_n, b_n]$. But from (ii), (iii) and (iv), we know that

$$\left.\begin{aligned} b(2^{n-1}\xi) &= b(\xi - 2\pi) \\ b(\xi) &= b\left(2^{n-1}(\xi - 2\pi)\right) \end{aligned}\right\} \quad \text{for a.e } \xi \in [e_n, b_n]. \tag{2.9}$$

So (2.8) can be written as

$$\left.\begin{aligned} \left[e^{i\alpha(\xi)}e^{i\theta(\xi)} + e^{-i\theta\left(2^{n-1}(\xi-2\pi)\right)}\right] b(\xi) &= 0 \\ \left[e^{i\alpha(\xi)}e^{i\theta(\xi-2\pi)} - e^{-i\theta(2^{n-1}\xi)}\right] b(2^{n-1}\xi) &= 0 \end{aligned}\right\} \quad \text{for a.e. } \xi \in [e_n, b_n].$$

This shows that

$$\begin{aligned} e^{-i\alpha(\xi)} &= e^{i[\theta(\xi)+\theta(2^{n-1}(\xi-2\pi))+\pi]} \quad \text{a.e. on } [e_n, b_n] \cap supp\ b. \\ e^{-i\alpha(\xi)} &= e^{i[\theta(\xi-2\pi)+\theta(2^{n-1}\xi)]} \quad \text{a.e. on } [e_n, b_n] \cap \left(\tfrac{1}{2^{n-1}} supp\ b\right). \end{aligned}$$

Hence, for almost every $\xi \in [e_n, b_n] \cap supp\ b \cap \left(\frac{1}{2^{n-1}} supp\ b\right)$, we have

$$\theta(\xi) + \theta\left(2^{n-1}(\xi - 2\pi)\right) - \theta(\xi - 2\pi) - \theta(2^{n-1}\xi) = (2m(\xi) + 1)\,\pi,$$

for some integer-valued measurable function m. This proves (v).

We now prove the converse. Suppose $\psi \in L^2(\mathbb{R})$, $supp\ \hat{\psi} \subseteq S_n$ and the function $b(\xi) = |\hat{\psi}(\xi)|$ satisfies conditions $(i) - (v)$ of the theorem. By Theorem 1.1 it is sufficient to show that $||\psi||_2 = 1$ and ψ satisfies (1.3) and (1.4). We have,

$$2\pi||\psi||_2^2 = ||\hat{\psi}||_2^2 = \int_{S_n} |\hat{\psi}(\xi)|^2\, d\xi =$$

$$
\begin{aligned}
&= \left(\int_{-d_n}^{-c_n} + \int_{-b_n}^{-e_n} + \int_{-e_n}^{-a_n} + \int_{a_n}^{e_n} + \int_{e_n}^{b_n} + \int_{c_n}^{d_n}\right) b^2(\xi)\, d\xi \\
&= \int_{-d_n}^{-c_n} + \int_{c_n}^{d_n} + \int_{-b_n}^{-e_n} + \int_{e_n}^{b_n} + 2(e_n - a_n).
\end{aligned}
$$

By changing variables $\xi \to 2^{n-1}(\xi - 2\pi), \xi \to 2^{n-1}\xi$ and $\xi \to \xi - 2\pi$ in the first, second and third integrals respectively, we get

$$
\begin{aligned}
2\pi\|\psi\|_2^2 &= 2^{n-1}\int_{e_n}^{b_n} \left[b^2\left(2^{n-1}(\xi - 2\pi)\right) + b^2(2^{n-1}\xi)\right] d\xi \\
&\quad + \int_{e_n}^{b_n} \left[b^2(\xi - 2\pi) + b^2(\xi)\right] d\xi + 2(e_n - a_n) \\
&= 2^{n-1}(b_n - e_n) + (b_n - e_n) + 2(e_n - a_n) = 2\pi,
\end{aligned}
$$

where we have used (2.5) and property (*iii*) of the theorem. Hence, $\|\psi\|_2 = 1$.

We will now show that ψ satisfies (1.3). Let

$$
\rho(\xi) = \sum_{j\in\mathbb{Z}} |\, \hat{\psi}(2^j\xi)\, |^2 .
$$

Suppose $\xi > 0$. Observe that

$$
\begin{aligned}
\mathbb{R}^+ &= \bigcup_{l\in\mathbb{Z}} 2^l[a_n, b_n] \\
&= \bigcup_{l\in\mathbb{Z}} 2^l\left([a_n, e_n] \cup [e_n, b_n]\right).
\end{aligned}
$$

So there is an $l \in \mathbb{Z}$ such that $\xi \in 2^l[a_n, e_n] \cup 2^l[e_n, b_n]$. If $\xi \in 2^l[a_n, e_n]$ then since $2^{-l}\xi \in [a_n, e_n]$, by Lemma 2.1 we have, $2^j(2^{-l}\xi) \notin S_n$ if $j \neq 0$. That is, $2^j\xi \notin S_n$ if $j \neq -l$. Hence, $\rho(\xi) = 1$, by (*i*). Similarly by using (*ii*), we can prove that $\rho(\xi) = 1$ if $\xi \in 2^l[e_n, b_n]$. A similar decomposition for $\xi < 0$ proves that $\rho(\xi) = 1$ for a.e. $\xi \in \mathbb{R}$.

Finally, we have to show that ψ satisfies (1.4). For $m \in 2\mathbb{Z} + 1$, let us denote the function on the left hand side of (1.4) by $t_m(\xi)$. Then

$$
\begin{aligned}
t_m(\xi) &= \sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}\left(2^j(\xi + 2m\pi)\right)} \\
&= \sum_{j\geq 0} \hat{\psi}\left(2^j(\xi + 2m\pi - 2m\pi)\right) \overline{\hat{\psi}\left(2^j(\xi + 2m\pi)\right)} \\
&= \overline{t_{-m}(\xi + 2m\pi)}.
\end{aligned}
$$

Therefore, we have only to show that $t_m(\xi) = 0$ for a.e. ξ, if $m \in 2\mathbb{Z}_- + 1$ (negative odd integers). Suppose $m \in 2\mathbb{Z}_- + 1$ and $m \neq -1$. Let $\xi \in \mathbb{R}$ and suppose that $2^j\xi \in S_n$. Since $j \geq 0$, $n \geq 2$ and $m \neq -1$ and odd, therefore $2^j m \neq 0, \pm 1, \pm 2^{n-1}$. So by Lemma 2.1, we have $2^j\xi + 2 \cdot 2^j m\pi \notin S_n$. This implies that each term of $t_m(\xi)$ is zero which proves that t_m is zero. It now remains to show that $t_{-1}(\xi) = 0$ for a.e ξ. We have

$$t_{-1}(\xi) = \sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j\xi - 2 \cdot 2^j\pi)}.$$

Let $\xi \in \mathbb{R}$ and suppose that $2^j\xi \in S_n$. Then again by Lemma 2.1, $2^j\xi - 2 \cdot 2^j\pi \notin S_n$ if $-2^j \neq 0, \pm 1, \pm 2^{n-1}$. So the only possible j's to contribute a non-zero term are $j = 0$ and $j = n-1$. Thus,

$$t_{-1}(\xi) = \hat{\psi}(\xi)\overline{\hat{\psi}(\xi - 2\pi)} + \hat{\psi}(2^{n-1}\xi)\overline{\hat{\psi}(2^{n-1}\xi - 2 \cdot 2^{n-1}\pi)}. \tag{2.10}$$

Now, both ξ and $\xi - 2\pi$ belong to S_n only if $\xi \in [e_n, b_n]$. Hence, the first term of (2.10) is zero unless $\xi \in [e_n, b_n]$. Similarly, both $2^{n-1}\xi$ and $2^{n-1}\xi - 2 \cdot 2^{n-1}\pi$ belong to S_n only when $2^{n-1}\xi \in [c_n, d_n]$, which is equivalent to saying that $\xi \in [e_n, b_n]$. That is, the second term of (2.10) is also zero unless $\xi \in [e_n, b_n]$. Thus we get, $t_{-1}(\xi) = 0$ if $\xi \notin [e_n, b_n]$. Now on $[e_n, b_n]$, we have, by (2.9)

$$t_{-1}(\xi) = b(\xi)b(2^{n-1}\xi)e^{i[\theta(\xi)-\theta(\xi-2\pi)]} + b(2^{n-1}\xi)b(\xi)e^{i[\theta(2^{n-1}\xi)+\theta(2^{n-1}(\xi-2\pi))]}.$$

If $\xi \notin [e_n, b_n] \cap supp\ b \cap (\frac{1}{2^{n-1}} supp\ b)$, then either $b(\xi) = 0$ or $b(2^{n-1}\xi) = 0$. So $t_{-1}(\xi) = 0$. And if $\xi \in [e_n, b_n] \cap supp\ b \cap (\frac{1}{2^{n-1}} supp\ b)$, then by (v), $t_{-1}(\xi) = 0$. This completes the characterization. □

There is another equation which characterizes all wavelets ψ such that $\hat{\psi}$ is supported in the set S_n and the function $b = |\hat{\psi}|$ is even. For $n = 2$, the following proposition is proved in [HW] (see Proposition 4.7, Chapter 3). We observe that the result can be extended to the general case.

Proposition 2.1. *Suppose that ψ is a wavelet of $L^2(\mathbb{R})$, $b = |\hat{\psi}|$, and $supp\ b \subseteq S_n$. Then b is almost everywhere even if and only if*

$$b^2(\xi) + b^2(2\pi - \xi) = 1 \quad for\ a.e.\ \xi \in [e_n, b_n]. \tag{2.11}$$

Proof: Suppose that b is an even function and $\xi \in [e_n, b_n]$ Since $-\xi \in [-b_n, -e_n]$, we have

$$\begin{aligned} 1 &= b^2(-\xi) + b^2(-\xi + 2\pi), \quad \text{by (2 3)} \\ &= b^2(\xi) + b^2(2\pi - \xi), \quad \text{since b is even,} \end{aligned}$$

which is (2 11)

Conversely, suppose that (2 11) holds Since by (i) of Theorem 2 1, b is even on $[a_n, e_n]$, it is enough to show that b is even on the sets $[e_n, b_n]$ and $[c_n, d_n]$

(a) Let $\xi \in [e_n, b_n]$ Therefore, $-\xi \in [-b_n, -e_n]$ Then

$$b^2(-\xi) + b^2(-\xi + 2\pi) = 1, \quad \text{by (2 3)}$$

This fact, together with (2 11), gives us

$$b(\xi) = b(-\xi) \tag{2 12}$$

(b) Now let $\xi \in [c_n, d_n]$ Therefore, $\frac{1}{2^{n-1}}\xi \in [e_n, b_n]$ From (2 12) we get

$$b\Big(\frac{1}{2^{n-1}}\xi\Big) = b\Big(-\frac{1}{2^{n-1}}\xi\Big) \tag{2 13}$$

Since $-\frac{1}{2^{n-1}}\xi \in [-b_n, -e_n]$, using (2 3) we obtain

$$b^2\Big(-\frac{1}{2^{n-1}}\xi\Big) + b^2\Big(-\frac{1}{2^{n-1}}\xi + 2\pi\Big) = 1 \tag{2 14}$$

Now, since $-\xi \in [-d_n, -c_n]$, we get (using (2 4))

$$b\Big(-\frac{1}{2^{n-1}}\xi + 2\pi\Big) = b(-\xi) \tag{2 15}$$

Substituting (2 13) and (2 15) in (2 14), we get

$$b^2\Big(\frac{1}{2^{n-1}}\xi\Big) + b^2(-\xi) = 1 \tag{2 16}$$

Comparing (2 16) and (2 2) we get $b(\xi) = b(-\xi)$ This proves the proposition $\square$

2.4 The wavelets associated with S_n, $n \geq 3$ are non-MRA

We will now show that, if $n \geq 3$, none of the wavelets characterized in Theorem 2 1 is associated with an MRA

Let ψ be any wavelet such that $\hat{\psi}$ is supported in S_n and let it be associated with an MRA Let ϕ and m_0 be the corresponding scaling function and low pass filter respectively Then we have (see equations (1 11) and (1 7))

$$|\hat{\phi}(\xi)|^2 = \sum_{j\geq 1} |\hat{\psi}(2^j\xi)|^2 \tag{2 17}$$

and

$$\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi) \tag{2 18}$$

Using (2 17) we can easily find that

$$|\hat{\phi}(\xi)| = \begin{cases} 1 & \text{if } |\xi| \leq a_n \\ b(2^{n-l-1}\xi) & \text{if } |\xi| \in 2^l[e_n, b_n], \quad 0 \leq l \leq n-2 \\ 0 & \text{otherwise} \end{cases} \tag{2 19}$$

That is,

$$|\hat{\phi}(2\xi)| = \begin{cases} 1 & \text{if } |\xi| \leq \frac{a_n}{2} \\ b(2^{n-l}\xi) & \text{if } |\xi| \in 2^{l-1}[e_n, b_n], \quad 0 \leq l \leq n-2 \\ 0 & \text{otherwise} \end{cases} \tag{2 20}$$

Case 1: $b(\xi) \not\equiv 1$ on $[e_n, b_n]$, i.e , $b(\xi) \not\equiv 0$ on $[c_n, d_n]$

If $|\xi| \leq a_n$, then from (2 18) and (2 19) we have

$$|\hat{\phi}(2\xi)| = |m_0(\xi)| \ |\hat{\phi}(\xi)| = |m_0(\xi)|$$

Therefore, by (2 20), we obtain

$$|m_0(\xi)| = 1 \quad \text{if } |\xi| \leq \frac{a_n}{2}$$

Since m_0 is 2π-periodic, we have

$$|m_0(\xi)| = 1 \quad \text{on } \left[-\frac{a_n}{2}, \frac{a_n}{2}\right] + 2 \ \ 2^{n-3}\pi$$

Note that $[-\frac{a_n}{2}, \frac{a_n}{2}] + 2 \ \ 2^{n-3}\pi = [\frac{c_n}{2}, \frac{d_n}{2}] = 2^{n-2}[e_n, b_n]$ Now, from (2 19), on $2^{n-2}[e_n, b_n]$ we have $|\hat{\phi}(\xi)| = b(2\xi)$ So $|\hat{\phi}(2\xi)| = |m_0(\xi)| \ |\hat{\phi}(\xi)| = b(2\xi)$ But by (2 20), $|\hat{\phi}(2\xi)| = 0$ on $2^{n-2}[e_n, b_n]$ Therefore, $b(2\xi) = 0$ on $2^{n-2}[e_n, b_n]$ That is, $b(\xi) = 0$ on $2^{n-1}[e_n, b_n] = [c_n, d_n]$, which is a contradiction

Case 2: $b(\xi) \equiv 1$ on $[e_n, b_n]$

As above, we have

$$|\hat{\phi}(\xi)| = \begin{cases} 1 & \text{if } |\xi| \le a_n \\ & \text{or if } \ \xi \in 2^l[-b_n, -e_n], \ \ 0 \le l \le n-2 \\ 0 & \text{otherwise} \end{cases} \tag{2 21}$$

Therefore,

$$|\hat{\phi}(2\xi)| = \begin{cases} 1 & \text{if } |\xi| \le \frac{a_n}{2} \\ & \text{or if } \ \xi \in 2^{l-1}[-b_n, -e_n], \ \ 0 \le l \le n-2 \\ 0 & \text{otherwise} \end{cases} \tag{2 22}$$

Now $|\hat{\phi}(2\xi)| = |m_0(\xi)| \ |\hat{\phi}(\xi)| = |m_0(\xi)|$ if $|\xi| \le a_n$, by (2 21) Therefore, by (2 22), $|m_0(\xi)| = 1$ if $|\xi| \le \frac{a_n}{2}$ Using the 2π-periodicity of m_0, we get

$$|m_0(\xi)| = 1 \quad \text{on } \left[-\frac{a_n}{2}, \frac{a_n}{2}\right] - 2 \ \ 2^{n-3}\pi$$

Note that

$$\left[-\frac{a_n}{2}, \frac{a_n}{2}\right] - 2 \cdot 2^{n-3}\pi = \left[-\frac{d_n}{2}, -\frac{c_n}{2}\right] = 2^{n-2}[-b_n, -e_n]$$

On $2^{n-2}[-b_n, -e_n]$, we have

$$|\hat{\phi}(2\xi)| = |m_0(\xi)| \ \ |\hat{\phi}(\xi)| = |m_0(\xi)| = 1$$

But by (2 22), $|\hat{\phi}(2\xi)| = 0$ on $[-\frac{d_n}{2}, -\frac{c_n}{2}]$, which is again a contradiction Therefore, the wavelets of Theorem 2 1 are non-MRA wavelets, if $n \ge 3$

2.5 The dimension functions of wavelets associated with S_n

Recall that for a wavelet ψ of $L^2(\mathbb{R})$ to be associated with an MRA, it is necessary and sufficient that $D_\psi = 1$ a e , where

$$D_\psi(\xi) = \sum_{j\geq 1}\sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi+2k\pi)\right)|^2$$

In this section we will compute the dimension functions for the wavelets characterized in Theorem 2 1 This will provide an alternative proof of the fact that these wavelets are not associated with any MRA These dimension functions are symmetric with respect to the origin and attain all integral values starting from 0 to the maximum value Also note that if $n \geq 3$, then all wavelets whose Fourier transform is supported in S_n have the same dimension function D_n We will prove the following proposition

Proposition 2.2. *Fix $n \geq 3$ Let ψ be a wavelet such that supp $\hat{\psi} \subseteq S_n$ and let D_n be the dimension function associated with ψ Then*

$$D_n(\xi) = \begin{cases} n-1 & a\ e.\ \ if\ |\xi| \in \left[0, \frac{2}{2^n-1}\pi\right] \\ r-1 & a\ e\ \ if\ |\xi| \in \left[\frac{2^{n-r}}{2^n-1}\pi, \frac{2^{n-r+1}}{2^n-1}\pi\right] \quad (2 \leq r \leq n-1) \\ 0 & a\ e\ \ if\ |\xi| \in \left[\frac{2^{n-1}}{2^n-1}\pi, \frac{2^n-2}{2^n-1}\pi\right] = [a_n, e_n] \\ 1 & a\ e\ \ if\ |\xi| \in \left[\frac{2^n-2}{2^n-1}\pi, \pi\right] = [e_n, \pi] \end{cases}$$

Proof: For $l \in \mathbb{Z}$, we define

$$p_l = \frac{2^{n+l-1}}{2^n-1}\pi = 2^l a_n \quad \text{and} \quad q_l = \frac{2^{n+l-1}-2^l}{2^n-1}\pi = 2^{l-1}e_n$$

Note that $p_0 = a_n$, $p_1 = 2p_0 = b_n$, $p_n = d_n$, $q_1 = e_n$ and $q_n = c_n$ Also observe that $p_l < q_{l+1} < p_{l+1}$ for all $l \geq 0$ As D_n is 2π-periodic, it is enough to compute its values for $\xi \in [-\pi, \pi]$ Again as $[-\pi, \pi] \subset [-p_1, p_1]$, we will compute D_n for the interval $[-p_1, p_1]$ Note that

$$(0, p_1] = \bigcup_{l\geq 0} 2^{-l}[p_0, p_1] = \bigcup_{l\geq 0}[p_{-l}, p_{-l+1}]$$

Case 1: $\xi \in [p_0, p_1]$

If $2^m \le k \le 2^{m+1}-1$ $(0 \le m \le n-3)$ then $\xi + 2k\pi \in [p_{n+2}, q_{m+3}]$ So $2^j(\xi+2k\pi) \in [p_{j+m+2}, q_{j+m+3}]$ If $j+m+3 \le n$ then $2^j(\xi+2k\pi) \in [p_1, q_n] = [b_n, c_n]$ which is not in $supp\ \hat{\psi}$, and if $j+m+3 \ge n+1$ then $2^j(\xi+2k\pi) \ge d_n$, which shows that $2^j(\xi+2k\pi) \notin supp\ \hat{\psi}$ Now if $k \ge 2^{n-2}$, then $\xi + 2k\pi \ge d_n$ Thus, we have proved that if $k \ge 1$, then $2^j(\xi+2k\pi) \notin supp\ \hat{\psi}$ for all $j \ge 1$ Similarly, we can show that if $k \le -2$, then $2^j(\xi+2k\pi) \notin supp\ \hat{\psi}$ for all $j \ge 1$ Therefore, we have only to consider $k = 0, -1$ Note that $[p_0, p_1] = [p_0, q_1] \cup [q_1, p_1]$

(a) If $\xi \in [p_0, q_1]$, then $2^j\xi \in [p_j, q_{j+1}]$ Now $j \ge n \Rightarrow 2^j\xi \ge p_n = d_n$ and $j \le n-1 \Rightarrow j+1 \le n \Rightarrow$ for all $j \ge 1$, $2^j\xi \in [p_j, q_n] \subset [p_1, q_n] = [b_n, c_n]$ which is not in $supp\ \hat{\psi}$ Also, $\xi - 2\pi \in [-q_2, -p_1] \Rightarrow 2^j(\xi - 2\pi) \in [-q_{j+2}, -p_{j+1}]$ A similar argument as in the case of $2^j\xi$ shows that $2^j(\xi-2\pi) \notin supp\ \hat{\psi}$ for all $j \ge 1$ So $D_n(\xi) = 0$ a e on $[p_0, p_1]$

(b) If $\xi \in [q_1, p_1]$, then $2^j\xi \in [q_{j+1}, p_{j+1}]$ So $2^j\xi \notin supp\ \hat{\psi}$ if $j \ne n-1$ Similarly, $2^j(\xi-2\pi) \in [-p_{j+1}, -q_{j+1}]$ So $2^j(\xi-2\pi) \notin supp\ \hat{\psi}$ if $j \ne n-1$ Therefore, $D_n(\xi) = |\hat{\psi}(2^{n-1}\xi)|^2 + |\hat{\psi}(2^{n-1}(\xi-2\pi))|^2 = 1$, by (2 6), as $2^{n-1}\xi \in [q_n, p_n] = [c_n, d_n]$

We now proceed to compute D_n on other sets Observe that

$$\begin{aligned} D_n(\xi) &= \sum_{k=-\infty}^{-1}\sum_{j\ge1}\left|\hat{\psi}\left(2^j(\xi+2k\pi)\right)\right|^2 + \sum_{j\ge1}\left|\hat{\psi}(2^j\xi)\right|^2 \\ &\quad + \sum_{k=1}^{\infty}\sum_{j\ge1}\left|\hat{\psi}\left(2^j(\xi+2k\pi)\right)\right|^2 \\ &= D_n^-(\xi) + D_n^0(\xi) + D_n^+(\xi), \text{ say} \end{aligned}$$

Let $\xi \in \left[\frac{2^{n-l}}{2^n-1}\pi, \frac{2^{n-l+1}}{2^n-1}\pi\right] = [p_{-l+1}, p_{-l+2}]$, $l \ge 2$

As in the above cases, we can show that if $k \le -2^{n-2}$, or if $-2^{m+1}-1 \le k \le -2^m-1$ $(0 \le m \le n-3)$, then $2^j(\xi+2k\pi) \notin supp\ \hat{\psi}$ for all $j \ge 1$ Also, if $k = -2^m (0 \le m \le n-3)$, then $2^j(\xi+2k\pi) \notin supp\ \hat{\psi}$ for all $j \ge 1$ unless $j = n-m-2$ Hence, in the sum $D_n^-(\xi)$, we have only to consider $j = n-m-2$ $(0 \le m \le n-3)$ and $k = -2^m$ Thus,

$$D_n^-(\xi) = \sum_{m=0}^{n-3}\left|\hat{\psi}\left(2^{n-m-2}(\xi - 2\ \ 2^m\pi)\right)\right|^2$$

In a similar manner, we can show that

$$D_n^+(\xi) = \sum_{m=0}^{n-3}\left|\hat{\psi}\left(2^{n-m-2}(\xi + 2 \cdot 2^m\pi)\right)\right|^2$$

Case 2: $\xi \in \left[\frac{2^{n-l}}{2^n-1}\pi, \frac{2^{n-l+1}}{2^n-1}\pi\right]$, $l \geq n$

By a straightforward calculation, one can show that if $l \geq n$, then

$$2^{n-m-2}(\xi + 2 \cdot 2^m\pi) \in [c_n, d_n], \quad \text{for} \quad 0 \leq m \leq n-3$$

Therefore,

$$\begin{aligned} & \hat{\psi}\left(2^{n-m-2}(\xi + 2 \cdot 2^m\pi)\right) + \hat{\psi}\left(2^{n-m-2}(\xi - 2 \cdot 2^m\pi)\right) \\ = \; & \hat{\psi}\left(2^{n-m-2}(\xi + 2 \cdot 2^m\pi)\right) + \hat{\psi}\left(2^{n-m-2}(\xi + 2 \cdot 2^m\pi) - 2^{n-1}\pi\right) \\ = \; & 1, \quad \text{by (2.6)} \end{aligned}$$

Hence, we get $D_n^+(\xi) + D_n^-(\xi) = n-2$. Now, $[p_{-l+1}, p_{-l+2}] = [p_{-l+1}, q_{-l+2}] \cup [q_{-l+2}, p_{-l+2}]$
(a) If $\xi \in [p_{-l+1}, q_{-l+2}]$, then $2^j\xi \notin supp\, \hat{\psi}$, if $j \neq l-1$. But $2^{l-1}\xi \in [p_0, q_1] = [a_1, e_n]$. Therefore, $D_n^0(\xi) = |\hat{\psi}(2^{l-1}\xi)|^2 = 1$.
(b) If $\xi \in [q_{-l+2}, p_{-l+2}]$, then $2^j\xi \notin supp\, \hat{\psi}$ if $j \neq l-1, n+l-2$. Now $2^{n+l-2}\xi \in [q_n, p_n] = [c_n, d_n]$. By (2.2) we get $D_n^0(\xi) = |\hat{\psi}(2^{n+l-2}\xi)|^2 + |\hat{\psi}(2^{l-1}\xi)|^2 = 1$. Thus, in either case $D_n(\xi) = n-2+1 = n-1$. We have proved that $D_n(\xi) = n-1$, if $\xi \in \left[\frac{2^{n-l}}{2^n-1}\pi, \frac{2^{n-l+1}}{2^n-1}\pi\right]$, $l \geq n$

Case 3: $\xi \in \left[\frac{2^{n-l}}{2^n-1}\pi, \frac{2^{n-l+1}}{2^n-1}\pi\right]$, $l = 2$

Again as in Case 2, it can be shown that

$$2^{n-m-2}(\xi - 2 \cdot 2^m\pi) \quad \text{and} \quad 2^{n-m-2}(\xi + 2 \cdot 2^m\pi)$$

are not in $supp\, \hat{\psi}$ for all m such that $0 \leq m \leq n-3$. Therefore,

$$D_n^-(\xi) = D_n^+(\xi) = 0$$

Now if $\xi \in \left[\frac{2^{n-2}}{2^n-1}\pi, \frac{2^{n-2}-1}{2^n-1}\pi\right]$ then $2^j\xi \notin supp\, \hat{\psi}$ if $j \neq 1$, and if $\xi \in \left[\frac{2^{n-2}-1}{2^n-1}\pi, \frac{2^{n-2}}{2^n-1}\pi\right]$ then $2^j\xi \notin supp\, \hat{\psi}$ if $j \neq 1, n$. In either case,

$$D_n(\xi) = D_n^0(\xi) = 1$$

Case 4: $3 \le l \le n-1$ (This case is required if $n \ge 4$.)

We can write $2^{n-m-2}\xi \in \left[\frac{2^{n+p-l}}{2^n-1}\pi, \frac{2^{n+p}}{2^n-1}\pi\right]$, where $p = n-m-l-1$. Using the condition on support of $\hat{\psi}$, we can show that if $\xi \in \left[\frac{2^{n+p-l}}{2^n-1}\pi, \frac{2^{n+p}}{2^n-1}\pi\right]$, then $\xi \pm 2 \cdot 2^{n-2}\pi \in supp\ \hat{\psi}$ only when $p \le -1$. So in order that $\hat{\psi}(2^{n-m-2}\xi \pm 2 \cdot 2^{n-2}\pi) \ne 0$, we must have $p = n-m-l-1 \le -1$, i.e., $m \ge n-l$. We also have, $0 \le m \le n-3$. Therefore, $1 \le n-m-2 \le l-2$. Thus

$$\begin{aligned} D_n(\xi) &= \sum_{j\ge 1}\left|\hat{\psi}(2^j\xi)\right|^2 \\ &\quad + \sum_{m=0}^{n-3}\left\{\left|\hat{\psi}\left(2^{n-m-2}(\xi+2 \cdot 2^m\pi)\right)\right|^2 + \left|\hat{\psi}\left(2^{n-m-2}(\xi-2 \cdot 2^m\pi)\right)\right|^2\right\} \\ &= \sum_{j\ge 1}\left|\hat{\psi}(2^j\xi)\right|^2 + \sum_{s=1}^{l-2}\left\{\left|\hat{\psi}(2^s\xi+2 \cdot 2^{n-2}\pi)\right|^2 + \left|\hat{\psi}(2^s\xi-2 \cdot 2^{n-2}\pi)\right|^2\right\} \end{aligned}$$

Now, $2^s\xi + 2 \cdot 2^{n-2}\pi \in [c_n, d_n]$, $1 \le s \le l-2$. By (2.6), the second sum on the right hand side of the last equality above is equal to $l-2$. Also as in the previous cases, the first sum can be shown to be equal to 1. Hence, $D_n(\xi) = l-1$. □

2.6 Concluding remarks

In [HKLS] MSF wavelets were considered where they were called unimodular wavelets. Let ψ be an MSF wavelet and $K = supp\ \hat{\psi}$. Define $K^+ = K \cap \mathbb{R}^+$ and $K^- = K \cap \mathbb{R}^-$. One of the results proved in [HKLS] is about wavelets ψ such that $K^- = -K^+$ and K^+ consists of two disjoint intervals. They proved the following.

(A) Let $\hat{\psi}(\xi) = \mu(\xi)\chi_K(\xi)$, where $K = K^+ \cup K^-, K^- = -K^+$ and $K^+ = [a_n, \pi] \cup [2^{n-1}\pi, d_n]$ with $|\mu(\xi)| = 1$. Then ψ is a wavelet for $L^2(\mathbb{R})$. Moreover, each MSF wavelet for which $K^- = -K^+$, and K^+ is a union of two disjoint intervals is of this form.

If we define $b = \chi_{[e_n,\pi]}$ on $[e_n, b_n]$, and extend it to S_n by using (i)–(iv) of Theorem 2.1, then we get the wavelets described in (A). In particular, for $n = 3$, the corresponding wavelet is the Journé wavelet.

The paper [HKLS] also contains the following result.

(B) Let $\hat{\psi}(\xi) = \mu(\xi)\chi_K(\xi)$ where

$$K^- = \left[-2\left(1 - \frac{2p+1}{2^n-1}\right)\pi, -\left(1 - \frac{2p+1}{2^n-1}\right)\pi\right]$$

and

$$K^+ = \left[\frac{2(p+1)}{2^n-1}\pi, \frac{2(2p+1)}{2^n-1}\pi\right] \cup \left[\frac{2^n(2p+1)}{2^n-1}\pi, \frac{2^{n+1}(p+1)}{2^n-1}\pi\right]$$

for $n \geq 3$, $1 \leq p \leq 2^{n-1}-2$ and $|\mu(\xi)| = 1$ Then ψ is a wavelet for $L^2(\mathbb{R})$ Moreover, each MSF wavelet for which K^- is an interval and K^+ is the union of two disjoint intervals is of this form

If we define $b(\xi) = 0$ on $[e_n, b_n]$ and extend it to S_n by using (i)-(iv) of Theorem 2 1, then we get

$$b = \chi_{[-b_n,-a_n]\cup[a_n,e_n]\cup[c_n,d_n]}$$

which corresponds to the case $p = 2^{n-2} - 1$ in (B) In particular, for $n = 3$, we get the Lemarié wavelet

Chapter 3

Some more band-limited wavelets of $L^2(\mathbb{R})$

In this chapter we will construct wavelets with interesting properties These wavelets will be generated with the help of the wavelets characterized in Chapter 2 We will construct two families of band-limited wavelets with Fourier transform discontinuous at the origin One of these families have Fourier transforms which are even functions We will also show that the equivalence classes of wavelets defined in [Web] are non-empty

3.1 Wavelets with discontinuous Fourier transform at the origin

Every wavelet ψ of $L^2(\mathbb{R})$ has to satisfy (1 3) Therefore, $\hat{\psi}(0)$ must be equal to 0 if $|\hat{\psi}|$ is continuous at 0 In fact, if ψ is a band-limited wavelet and $|\hat{\psi}|$ is continuous at 0, then a stronger result holds $|\hat{\psi}| = 0$ *a e in an open neighbourhood of the origin* (see [HW, Theorem 2 7, Chapter 3]) In [HWW1], it was shown that if ψ is a wavelet with *supp* $\hat{\psi} \subset [-4\pi + \frac{4}{3}\alpha, \frac{8}{3}\alpha], 0 < \alpha \leq \pi$, then $|\hat{\psi}| = 0$ on the interval $[-2\pi + \frac{4}{3}\alpha, \frac{2}{3}\alpha]$ It is interesting to observe the behaviour of $\hat{\psi}$ at the origin if we enlarge the support of $\hat{\psi}$ Some related results are the following

(1) Madych [Mad] constructed an example of a band-limited scaling function so that the corresponding wavelet ψ (which is also band-limited) has the property that $\hat{\psi}$ does not vanish in any neighbourhood of the origin

(2) Garrigós [Gar], in his Ph D thesis, constructed a wavelet ψ with *supp* $\hat{\psi} \subset [-4\pi, \pi]$ and $\hat{\psi}$ not vanishing in any neighbourhood of the origin

(3) In [BGRW], the authors constructed for each $\epsilon > 0$, a wavelet ψ_ϵ such that *supp* $\hat{\psi}_\epsilon \subset [-\frac{8}{3}\pi, \frac{8}{3}\pi + \epsilon]$, but $\hat{\psi}_\epsilon$ does not vanish in any neighbourhood of the origin

Recall that (see Definition 1.2, Chapter 1) a wavelet ψ is called a minimally supported frequency (MSF) wavelet if $|\hat{\psi}| = \chi_K$ for some measurable subset K of $\mathbb{R}$. Such a set K is called a wavelet set, which necessarily has measure 2π

In §3 1 1 we will construct a family of wavelets with Fourier transform not vanishing in any neighbourhood of the origin The wavelets we are going to construct (as well as the wavelets mentioned in (1)-(3) above) are MSF wavelets so that it is sufficient to construct the associated wavelet sets For $n \geq 2$, we will start with a wavelet set W (which depends on n) and construct another set W_n Then we will show that W_n is both dilation and translation equivalent to the wavelet set W, which will prove, by Corollary 1 2 of Chapter 1, that W_n is also a wavelet set

3.1.1 A family of band-limited wavelets with Fourier transform discontinuous at the origin

Let $n \geq 2$ Put

$$\begin{aligned} a_n &= \tfrac{2^{n-1}}{2^n-1}\pi, & b_n &= 2a_n = \tfrac{2^n}{2^n-1}\pi, \\ c_n &= \tfrac{2^{n-1}(2^n-2)}{2^n-1}\pi, & d_n &= 2^n a_n = \tfrac{2^{2n-1}}{2^n-1}\pi, \\ e_n &= \tfrac{2^n-2}{2^n-1}\pi \end{aligned}$$

Let

$$L = [-d_n, -c_n], \quad M = [-e_n, -a_n] \quad \text{and} \quad R = [a_n, b_n]$$

In Theorem 2 1, by taking $b(\xi) = 1$ on $[e_n, b_n]$ and extending to the whole of S_n, we get a wavelet whose Fourier transform is the characteristic function of the set $L \cup M \cup R$

Therefore, $W = L \cup M \cup R$ is a wavelet set Let ϵ be a positive number such that $\epsilon < \frac{a_n}{2}$ Define the following sets

$$\begin{aligned} P_1 &= \left[\frac{a_n}{2} + \frac{\epsilon}{2^n}, \frac{a_n}{2} + \epsilon\right], \\ P_2 &= [a_n + 2\epsilon, b_n], \\ P_3 &= [d_n, d_n + 2\epsilon] \end{aligned}$$

To make P_2 a non-empty set we need $2\epsilon < a_n$ Let

$$\begin{aligned} &X_0 = P_1 - 2 \ 2^{n-2}\pi, \quad Y_0 = \tfrac{1}{2^n} X_0, \\ &X_l = Y_{l-1} - 2 \cdot 2^{n-2}\pi, \quad Y_l = \tfrac{1}{2^{n+l}} X_l, \quad l \geq 1 \end{aligned}$$

Now define the set

$$W_n = \left(L \setminus \bigcup_{l=0}^{\infty} X_l\right) \cup \left(\bigcup_{l=0}^{\infty} Y_l\right) \cup M \cup (P_1 \cup P_2 \cup P_3) \tag{3 1}$$

Theorem 3.1. *The set W_n, defined in (3 1), is a wavelet set for $n \geq 2$*

Proof: We will prove the theorem by showing that W_n is translation and dilation equivalent to the wavelet set W First of all, we show by induction that $X_l \subset L$ for all $l \geq 0$ Note that $P_1 \subset [-a_n, a_n]$ So

$$X_0 = P_1 - 2 \ 2^{n-2}\pi \subset [-a_n, a_n] - 2^{n-1}\pi = [-d_n, -c_n] = L$$

Now, assume that $X_m \subset L$ Then,

$$\begin{aligned} &Y_m = \frac{1}{2^{m+n}} X_m \subset \left[-\frac{d_n}{2^{m+n}}, -\frac{c_n}{2^{m+n}}\right] \subset [-a_n, a_n] \\ &\Longrightarrow X_{m+1} = Y_m - 2^{n-1}\pi \subset L \end{aligned}$$

The interval X_0 lies in the interval $[-2^{n-1}\pi, -c_n]$ and $\{X_l \ \ l \geq 1\}$ lie in $[-d_n, -2^{n-1}]$ Further, X_{l+1} lies to the right of $X_l, l \geq 1$ The intervals Y_l, $l \geq 0$ lie in $\frac{1}{2^l}[-a_n, -\frac{a_n}{2}]$ so that Y_{l+1} lies to the right of Y_l for all $l \geq 0$ Also observe that $\{X_l, Y_l \ \ l \geq 0\}$ is a disjoint collection

Dilation equivalence

$$2P_1 \cup P_2 \cup \frac{1}{2^n} P_3 = \left[a_n + \frac{\epsilon}{2^{n-1}}, a_n + 2\epsilon\right] \cup [a_n + 2\epsilon, b_n] \cup \left[a_n, a_n + \frac{\epsilon}{2^{n-1}}\right]$$

$$= \quad [a_n, b_n] = R,$$

$$\left(L \setminus \bigcup_{l=0}^{\infty} X_l\right) \cup \left(\bigcup_{l=0}^{\infty} 2^{n+l} Y_l\right) = \left(L \setminus \bigcup_{l=0}^{\infty} X_l\right) \cup \left(\bigcup_{l=0}^{\infty} X_l\right) = L$$

(since $X_l \subset L$, for all $l \geq 0$) The set M appears in both the partitions of W and W_n

Translation equivalence

$$P_2 \cup (P_3 - 2 \quad 2^{n-2}\pi) = [a_n + 2\epsilon, b_n] \cup [a_n, a_n + 2\epsilon] = R,$$

$$\begin{aligned}
&\left(L \setminus \bigcup_{l=0}^{\infty} X_l\right) \cup \left(\bigcup_{l=0}^{\infty} (Y_l - 2 \quad 2^{n-1}\pi)\right) \cup (P_1 - 2 \quad 2^{n-2}\pi) \\
&= \quad \left(L \setminus \bigcup_{l=0}^{\infty} X_l\right) \cup \left(\bigcup_{l=1}^{\infty} X_l\right) \cup X_0 \\
&= \quad L
\end{aligned}$$

Again, M appears in both the partitions of W and W_n As W_n is translation and dilation equivalent to the wavelet set W, we have proved that W_n is a wavelet set □

Let $\hat{\psi}_n$ be the characteristic function of W_n Then ψ_n is a band-limited wavelet Since $Y_l \subset \frac{1}{2^l}\left[-a_n, -\frac{a_n}{2}\right]$ for all $l \geq 0$, $\hat{\psi}_n$ does not vanish in any neighbourhood of 0 In particular, $\hat{\psi}_n$ is discontinuous at the origin

3.1.2 Band-limited wavelets with Fourier transform even and discontinuous at the origin

In [Gar] the question of the existence of a wavelet with the following three properties was asked

(i) ψ is band-limited, i e., $\hat{\psi}$ is compactly supported,

(ii) $\hat{\psi}$ is even, and

(iii) $\hat{\psi}$ does not vanish in any neighbourhood of the origin

Examples of wavelets satisfying any two of the above three properties can be constructed The Shannon wavelet ψ (where $\hat{\psi} = \chi_{[-2\pi,\pi]\cup[\pi,2\pi]}$) satisfies (i) and (ii) but not (iii) The wavelets constructed in §3 1 1, as well as the wavelets of [Mad], [Gar] and [BGRW] referred in §3 1 1, satisfy (i) and (iii) but not (ii) N Arcozzi [Arc] constructed a wavelet which satisfies (ii) and (iii) but not (i) In this section we will construct a family of wavelets satisfying all the three properties listed above These wavelets are again MSF wavelets and so we construct the associated wavelet sets To get these wavelets we will suitably modify the method presented in §3 1 1

For $n \geq 2$, let a_n, b_n, d_n and e_n be as in §3 1 1

Let

$$L_1 = [-\pi, -a_n], \quad L_2 = [-d_n, -2^{n-1}\pi],$$

and

$$R_1 = [a_n, \pi], \quad R_2 = [2^{n-1}\pi, d_n]$$

On $[e_n, b_n]$, define $b = \chi_{[e_n,\pi]}$ Then, by extending b to the whole of S_n (see Theorem 2 1), we see that the characteristic function of the set $L_1 \cup L_2 \cup R_1 \cup R_2$ is the Fourier transform of a wavelet Therefore, it is a wavelet set For $\epsilon > 0$ such that $\epsilon < \frac{1}{4}\left(\frac{2^n-2}{2^n-1}\pi\right) = \frac{e_n}{4}$, construct the following sets·

$$\begin{aligned} S_1 &= \left[\frac{a_n}{2} + \frac{\epsilon}{2^n}, \frac{a_n}{2} + \epsilon\right], \\ S_2 &= [a_n + 2\epsilon, \pi], \\ S_3 &= [d_n, d_n + 2\epsilon] \end{aligned}$$

and

$$T_i = -S_i \quad \text{for } i = 1, 2, 3$$

To make S_2 a non-empty set, we need to take $a_n + 2\epsilon < \pi$ which is equivalent to $\epsilon < \frac{e_n}{4}$

Let

$$\begin{aligned} E_0 &= S_1 + 2\ 2^{n-2}\pi, \quad F_0 = \tfrac{1}{2^{n+1}} E_0, \\ E_l &= F_{l-1} + 2\ 2^{n-2}\pi, \quad F_l = \tfrac{1}{2^{n+l+1}} E_l, \quad l \geq 1, \\ G_l &= -E_l \quad \text{and} \quad H_l = -F_l, \quad \text{for } l \geq 0 \end{aligned}$$

Define

$$K_n = \left(R_2 \setminus \bigcup_{l=0}^{\infty} E_l\right) \cup \left(\bigcup_{l=0}^{\infty} F_l\right) \cup (S_1 \cup S_2 \cup S_3)$$
$$\cup\left(L_2 \setminus \bigcup_{l=0}^{\infty} G_l\right) \cup \left(\bigcup_{l=0}^{\infty} H_l\right) \cup (T_1 \cup T_2 \cup T_3) \tag{3 2}$$

Theorem 3.2. *For each $n \geq 2$, the set K_n, defined in* (3 2), *is a wavelet set*

Proof: We will prove that K_n is translation and dilation equivalent to the wavelet set $L_1 \cup L_2 \cup R_1 \cup R_2$ As in §3 1 1, we can show by induction that $E_l \subset R_2$ for all $l \geq 0$ By symmetry of the set $K_n, G_l \subset L_2$ for all $l \geq 0$ The intervals E_l, $l \geq 0$ lie inside the interval $[2^{n-1}\pi, d_n]$ and E_{l+1} lies to the left of E_l for all $l \geq 0$ Similarly, the intervals F_l, $l \geq 0$ lie in $\frac{1}{2^l}\left[\frac{a_n}{4}, \frac{a_n}{2}\right]$ so that F_{l+1} lies to the left of F_l for $l \geq 0$ Similar statements are true for the intervals G_l and F_l, $l \geq 0$

Dilation equivalence

$$\begin{aligned} 2S_1 \cup S_2 \cup \frac{1}{2^n} S_3 &= \left[a_n + \frac{\epsilon}{2^{n-1}}, a_n + 2\epsilon\right] \cup [a_n + 2\epsilon, \pi] \cup \left[a_n, a_n + \frac{\epsilon}{2^{n-1}}\right] \\ &= [a_n, \pi] = R_1 \end{aligned}$$

and

$$\left(R_2 \setminus \bigcup_{l=0}^{\infty} E_l\right) \cup \left(\bigcup_{l=0}^{\infty} 2^{n+l+1} F_l\right) = \left(R_2 \setminus \bigcup_{l=0}^{\infty} E_l\right) \cup \left(\bigcup_{l=0}^{\infty} E_l\right) = R_2$$

(since $E_l \subset R_2$, for all $l \geq 0$) Similarly for L_1 and L_2

Translation equivalence

$$S_2 \cup (S_3 - 2\ \ 2^{n-2}\pi) = [a_n + 2\epsilon, \pi] \cup [a_n, a_n + 2\epsilon] = R_1$$

and

$$\begin{aligned} &\left(R_2 \setminus \bigcup_{l=0}^{\infty} E_l\right) \cup \left(\bigcup_{l=0}^{\infty} (F_l + 2\ \ 2^{n-2}\pi)\right) \cup (S_1 + 2\ \ 2^{n-2}\pi) \\ &= \left(R_2 \setminus \bigcup_{l=0}^{\infty} E_l\right) \cup \left(\bigcup_{l=1}^{\infty} E_l\right) \cup E_0 = R_2 \end{aligned}$$

Similarly for L_1 and L_2 Thus, we have shown that K_n is both dilation and translation equivalent to the wavelet set $L_1 \cup L_2 \cup R_1 \cup R_2$, and hence is a wavelet set □

Let $\hat{w}_n$ be the characteristic function of the set K_n So w_n is a band-limited wavelet such that $\hat{w}_n$ is even Since $F_l \subset \frac{1}{2^l}[\frac{\pi}{4}, \frac{a_n}{2}]$ for all $l \geq 0$, $\hat{w}_n$ does not vanish in any neighbourhood of 0 In particular, it is discontinuous at 0

3.2 Dimension functions of band-limited wavelets

Let us recall that for every wavelet ψ, there is an associated function D_ψ, called the dimension function

$$D_\psi(\xi) = \sum_{j\geq 1} \sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi + 2k\pi)\right)|^2$$

It is clear that D_ψ is 2π-periodic and is in $L^2(\mathbb{T})$ First of all, let us observe that the dimension function is bounded for every band-limited wavelet

Proposition 3.1. *If ψ is a wavelet and supp $\hat{\psi} \subset [-2n\pi, 2n\pi]$ for some $n \in \mathbb{N}$, then $D_\psi \leq n$*

Proof: Let

$$F(\xi) = \sum_{j\geq 1} |\hat{\psi}(2^j\xi)|^2$$

By the condition on *supp* $\hat{\psi}$, we have *supp* $F \subset [-n\pi, n\pi]$ Also, since $\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$ for every wavelet ψ (see (1 3)), we observe that $F \leq 1$ Therefore, $F \leq \chi_{[-n\pi,n\pi]}$ This implies that

$$D_\psi(\xi) = \sum_{k\in\mathbb{Z}} F(\xi + 2k\pi) \leq \sum_{k\in\mathbb{Z}} \chi_{[-n\pi,n\pi]}(\xi + 2k\pi) = n,$$

which proves the proposition □

The statement of the above proposition is not sharp It was proved in [HW] (see also [HWW1]) that if *supp* $\hat{\psi} \subset \left[-\frac{8}{3}\pi, \frac{8}{3}\pi\right]$, then ψ is an MRA wavelet, i e , $D_\psi = 1$ a e Now consider the MSF wavelet constructed in §3 1 1 for the case $n = 2$ The corresponding wavelet set is W_2, as defined in (3 1) If $\xi \in [\frac{4}{3}\pi, \frac{4}{3}\pi + \epsilon]$, then

$$4(\xi - 2\pi) \in \left[-\frac{8}{3}\pi, -\frac{8}{3}\pi + 4\epsilon\right] \quad \text{and} \quad 2\xi \in \left[\frac{8}{3}\pi, \frac{8}{3}\pi + 2\epsilon\right]$$

Both the sets $[-\frac{8}{3}\pi, -\frac{8}{3}\pi + 4\epsilon]$ and $[\frac{8}{3}\pi, \frac{8}{3}\pi + 2\epsilon]$ are contained in the wavelet set W_2 Therefore, on the set $[\frac{4}{3}\pi, \frac{4}{3}\pi + \epsilon]$, the dimension function is at least 2 (the fact that it is at most 2 will follow from the proposition we prove below) So by increasing the support of $\hat{\psi}$ slightly larger than $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$, we no longer have the dimension function of ψ bounded by 1 That is, $\left[-\frac{8}{3}\pi, \frac{8}{3}\pi\right]$ is the largest symmetric interval to conclude that the dimension function is bounded by 1 whenever $\hat{\psi}$ is supported in this interval This motivates to find out other symmetric intervals of the form $[-t_n, t_n], n \geq 2$ such that *supp* $\hat{\psi} \subseteq [-t_n, t_n]$ implies that D_ψ is bounded by n The following result provides a lower bound for t_n

Proposition 3.2. *If ψ is a wavelet and supp $\hat{\psi} \subset [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi]$ for some $n \in \mathbb{N}$, then $D_\psi \leq n$*

Proof: Since D_ψ is 2π-periodic, we will prove that if ψ satisfies the hypothesis, then $D_\psi(\xi) \leq n$ for $\xi \in [-\pi, \pi]$ For $\xi \in [-\pi, \pi]$, we have $2^j(2k-1)\pi \leq 2^j(\xi + 2k\pi) \leq 2^j(2k+1)\pi$

(i) $j = n$ If $k \geq 2$, then $2^j(\xi + 2k\pi) = 2^n(\xi + 2k\pi) \geq 2^n(2k-1)\pi \geq 3 \cdot 2^n\pi \geq \frac{2^{n+2}}{3}\pi$ Similarly, if $k \leq -2$, then $2^n(\xi + 2k\pi) \leq -\frac{2^{n+2}}{3}\pi$ Hence, for $j = n$, the only non-zero terms contributing to D_ψ are for $k = -1, 0, 1$

(ii) $j \geq n+1$ If $k \geq 1$, then $2^j(\xi + 2k\pi) \geq 2^j(2k-1)\pi \geq 2^{n+1}(2k-1)\pi \geq 2^{n+1}\pi \geq \frac{2^{n+2}}{3}\pi$ Similarly, if $k \leq -1$, then $2^j(\xi + 2k\pi) \leq \frac{2^{n+2}}{3}\pi$ Hence, for $j \geq n+1$, only contributing k to D_ψ is $k = 0$

Thus, we have

$$\begin{aligned} D_\psi(\xi) &= \sum_{j=1}^{n-1}\sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi + 2k\pi)\right)|^2 \\ &\quad + \sum_{k=-1}^{1} |\hat{\psi}(2^n(\xi + 2k\pi))|^2 + \sum_{j\geq n+1} |\hat{\psi}(2^j\xi)|^2 \qquad (3\,3) \\ &= \sum_{j=1}^{n-1}\sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi + 2k\pi)\right)|^2 \\ &\quad + \left\{|\hat{\psi}\left(2^n(\xi - 2\pi)\right)|^2 + |\hat{\psi}\left(2^n(\xi + 2\pi)\right)|^2\right\} + \sum_{j\geq n} |\hat{\psi}(2^j\xi)|^2 \qquad (3\,4) \end{aligned}$$

If $\xi \in [-\frac{2}{3}\pi, \frac{2}{3}\pi]$, then $2^n(\xi + 2\pi) \geq 2^n \cdot \frac{4}{3}\pi = \frac{2^{n+2}}{3}\pi$ Similarly, $2^n(\xi - 2\pi) \leq -\frac{2^{n+2}}{3}\pi$ So both the terms in the second summand in the right hand side of (3 4) are zero and $D_\psi \leq n - 1 + 1 = n$, where we have used equations (1 3) and (1 5) Now, if $\xi \in [\frac{2}{3}\pi, \pi]$, then for all $j \geq n+1$, $2^j\xi \geq 2^{n+1} \cdot \frac{2}{3}\pi = \frac{2^{n+2}}{3}\pi$ So the last sum in (3 3) is zero and again $D_\psi \leq n$ In a similar manner, it can be shown that $D_\psi \leq n$ if $\xi \in [-\pi, -\frac{2}{3}\pi]$ This finishes the proof of the proposition □

Proposition 3 2 was also proved by Z Rzeszotnik and D Speegle in an unpublished article In the same article they also proved that for every $n \geq 2$ and $0 < \epsilon < \delta(n)$, there exists a wavelet ψ with Fourier transform supported in the set $[-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ such that $D_\psi > n$ on a set of positive measure This shows that $[-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi]$ is the largest symmetric interval to conclude that the dimension function D_ψ is bounded by n whenever $\hat{\psi}$ is supported in this interval We are grateful to Prof Guido Weiss for providing this information to us

3.3 Equivalence classes of wavelets

First of all, we will recall the equivalence relation on the set of all wavelets of $L^2(\mathbb{R})$ defined in [Web] Let ψ be a wavelet of $L^2(\mathbb{R})$ Define $V_j = \overline{sp}\{\psi_{l,k} : l < j,\ k \in \mathbb{Z}\}$ Then, it is easy to verify that the subspaces V_j, $j \in \mathbb{Z}$ satisfy the following properties

1 $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$

2 $f \in V_j$ if and only if $f(2\,\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$

3 $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\cap_{j\in\mathbb{Z}} = \{0\}$

4 V_0 is invariant under translation by integers

A natural question is Are there other groups of translations under which V_0 is invariant? For $\alpha \in \mathbb{R}$, let T_α denote the unitary operator $T_\alpha f = f(\cdot - \alpha)$ Consider the group of

translations $\mathcal{G}_n = \{T_{\frac{m}{2^n}} \; m \in \mathbb{Z}\}$ and the group $\mathcal{G}_\infty = \{T_\alpha \; \alpha \in \mathbb{R}\}$ Let $\mathcal{L}_n$ be the collection of all wavelets such that the corresponding space V_0 is invariant under the group $\mathcal{G}_n$ Then we have

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdot \quad \mathcal{L}_n \supset \mathcal{L}_{n+1} \supset \cdot \quad \supset \mathcal{L}_\infty$$

An equivalence relation can now be defined on the collection of wavelets The equivalence classes are given by $\mathcal{M}_n = \mathcal{L}_n \setminus \mathcal{L}_{n+1}$, with $\mathcal{M}_\infty = \mathcal{L}_\infty$ Therefore, $\mathcal{M}_n$ is the class of wavelets such that V_0 is invariant under the group $\mathcal{G}_n$ but not under $\mathcal{G}_{n+1}$

In [Web] Weber proved the following facts

(i) $\mathcal{M}_\infty = \cap_{n\geq 0}\mathcal{L}_n$

(ii) $\mathcal{M}_\infty$ is precisely the collection of all MSF wavelets

For the classes $\mathcal{M}_n, n \geq 1$ the following characterization is given

(iii) The equivalence class $\mathcal{M}_n$ consists of all wavelets ψ such that *supp* $\hat{\psi}$ is not partially self similar with respect to any odd multiple of $2^k\pi$, $k = 1, 2, \quad , n$ but is partially self similar with respect to some odd multiple of $2^{n+1}\pi$

Definition 3.1. [Web] *A set E is said to be partially self similar with respect to $\alpha \in \mathbb{R}$ if there is a set F of positive measure such that both F and $F + \alpha$ are subsets of E*

In the same paper, Weber produced examples of wavelets belonging to the first few equivalence classes $\mathcal{M}_n$, namely for $n = 0, 1, 2$ and 3 This motivated us to construct wavelets belonging to each equivalence class $\mathcal{M}_n$ After we constructed these wavelets we became aware of the article [SW], in which S Schaffer and E Weber constructed wavelets for the classes $\mathcal{M}_n, n \geq 1$ by using the method of "operator interpolation" (see [DL]) In fact, they did this for other dilation factors as well In this section we will construct wavelets belonging to each of the classes $\mathcal{M}_n, n \geq 0$ by a different method Our approach will be simpler than that of [SW] in the sense that for each $n \geq 3$, we will construct a function ψ_n such that ψ_n has the required properties to be in $\mathcal{M}_{n-2}$ as characterized by

Weber Then we will show that ψ_n is a wavelet Further, we will construct a family of wavelets belonging to the class $\mathcal{M}_0$

Note that, in view of the characterization of $\mathcal{M}_n$, to prove that $\psi_n \in \mathcal{M}_{n-2}$, it is sufficient to show that ψ_n satisfies the following two conditions

(a) $(supp\ \hat{\psi}_n) \cap (supp\ \hat{\psi}_n + 2^k q\pi) = \emptyset$, for all $q \in 2\mathbb{Z}+1$ and $k = 1, 2, \quad n-2$

(b) There exist a subset E of $supp\ \hat{\psi}_n$ such that $E + 2^{n-1}q\pi \subset supp\ \hat{\psi}_n$ for some $q \in 2\mathbb{Z}+1$

3.3.1 Construction of wavelets in $\mathcal{M}_n$, $n \geq 1$

Let $n \geq 3$ and define a_n, b_n, c_n, d_n and e_n as in Chapter 2 That is,

$$\begin{aligned} a_n &= \tfrac{2^{n-1}}{2^n-1}\pi, \qquad b_n = 2a_n = \tfrac{2^n}{2^n-1}\pi, \\ c_n &= \tfrac{2^{n-1}(2^n-2)}{2^n-1}\pi, \quad d_n = \tfrac{2^{2n-1}}{2^n-1}\pi, \\ e_n &= \tfrac{2^n-2}{2^n-1}\pi \end{aligned}$$

Our starting point is the MSF wavelet η given via its Fourier transform $\hat{\eta} = \chi_{W_n}$, where

$$W_n = [-b_n, -a_n] \cup [a_n, e_n] \cup [c_n, d_n]$$

The wavelet η belongs to the class of wavelets characterized in Theorem 2 1 of Chapter 2 (see (B) of §2 6)

We will construct ψ_n from this function in the following manner We translate $[\frac{a_n}{2}, \frac{e_n}{2}] + 2^{n-1}\pi \subset [c_n, d_n]$ to the left by a factor of $2^{n-1}\pi$ and assign values $\frac{1}{\sqrt{2}}$ to $\hat{\psi}_n$ on both these sets Then, we translate $[a_n, e_n]$ to the right by a factor of $2^n\pi$ and assign to $\hat{\psi}_n$ the value $\frac{1}{\sqrt{2}}$ on $[a_n, e_n]$ and $-\frac{1}{\sqrt{2}}$ on $[a_n, e_n] + 2^n\pi$ We leave the remaining sets of W_n as it is More precisely, we have the following function

$$\hat{\psi}_n(\xi) = \begin{cases} 1 & \text{if } \xi \in [-b_n, -a_n] \cup [c_n, \frac{a_n}{2} + 2^{n-1}\pi] \cup [\frac{e_n}{2} + 2^{n-1}\pi, d_n] \\ \frac{1}{\sqrt{2}} & \text{if } \xi \in [\frac{a_n}{2}, \frac{e_n}{2}] \cup [a_n, e_n] \cup [\frac{a_n}{2} + 2^{n-1}\pi, \frac{e_n}{2} + 2^{n-1}\pi] \\ -\frac{1}{\sqrt{2}} & \text{if } \xi \in [a_n + 2^n\pi, e_n + 2^n\pi] \\ 0 & \text{otherwise} \end{cases}$$

Let us make some observations about the set $F_n = supp\ \hat{\psi}_n$ It is easy to verify the following facts (stated in a tabular form for easy reference)

$\xi \in$ the set	$\xi + 2k\pi \notin F_n$ unless	$2^j\xi \notin F_n$ unless
$[-b_n, -e_n]$	$k = 0$	$j = 0$
$[-e_n, -a_n]$	$k = 0$	$j = 0$
$[\frac{a_n}{2}, \frac{e_n}{2}]$	$k = 0, 2^{n-2}$	$j = 0, 1$
$[a_n, e_n]$	$k = 0, 2^{n-1}$	$j = 0, -1$
$[c_n, \frac{a_n}{2} + 2^{n-1}\pi]$	$k = 0$	$j = 0$
$[\frac{a_n}{2} + 2^{n-1}\pi, \frac{e_n}{2} + 2^{n-1}\pi]$	$k = 0, -2^{n-2}$	$j = 0, 1$
$[\frac{e_n}{2} + 2^{n-1}\pi, d_n]$	$k = 0$	$j = 0$
$[a_n + 2^n\pi, e_n + 2^n\pi]$	$k = 0, -2^{n-1}$	$j = 0, -1$

Theorem 3.3. *For each $n \geq 3$, the function ψ_n defined above is a wavelet and belongs to the equivalence class $\mathcal{M}_{n-2}$*

Proof: To prove that ψ_n is a wavelet, it is sufficient to show that ψ satisfies the following three properties (see Theorem 1.1)

(1) $\|\psi_n\|_2 = 1$

(2) $H(\xi) \ = \sum\limits_{j\in\mathbb{Z}} |\hat{\psi}_n(2^j\xi)|^2 = 1$ for a e $\xi \in \mathbb{R}$

(3) $t_q(\xi) \ = \sum\limits_{j\geq 0} \hat{\psi}_n(2^j\xi)\overline{\hat{\psi}_n\left(2^j(\xi+2q\pi)\right)} = 0$ for a e $\xi \in \mathbb{R}$ and for all $q \in 2\mathbb{Z}+1$

Proof of (1)

We have

$$\begin{aligned}\|\hat{\psi}_n\|_2^2 &= (b_n - a_n) + \left(\frac{a_n}{2} + 2^{n-1}\pi - c_n\right) + d_n - \left(\frac{e_n}{2} + 2^{n-1}\pi\right) \\ &+ \frac{1}{2}\left\{\left(\frac{e_n}{2} - \frac{a_n}{2}\right) + (e_n - a_n) + \left(\frac{e_n}{2} - \frac{a_n}{2}\right) + (e_n - a_n)\right\}\end{aligned}$$

$$= b_n - a_n + d_n - c_n + e_n - a_n = 2\pi$$

Therefore, $\|\psi_n\|_2 = 1$

Proof of (2)

Since $H(\xi) = H(2\xi)$, it is enough to show that $H(\xi) = 1$ for a e $\xi \in [\alpha, 2\alpha]$ for some $\alpha > 0$ We will prove that $H(\xi) = 1$ for a e $\xi \in [a_n, 2a_n] = [a_n, b_n] = [a_n, e_n] \cup [e_n, b_n]$

Suppose $\xi \in [a_n, e_n]$ Then $2^j\xi \in F_n$ only when $j = 0, -1$ So, $H(\xi) = |\hat{\psi}(\xi)|^2 + |\hat{\psi}(\frac{\xi}{2})|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$ Now,

$$\begin{aligned}
\xi \in [e_n, b_n] \Leftrightarrow 2^{n-1}\xi &\in [c_n, d_n] \\
&= \left[c_n, \frac{a_n}{2} + 2^{n-1}\pi\right] \cup \left[\frac{a_n}{2} + 2^{n-1}\pi, \frac{e_n}{2} + 2^{n-1}\pi\right] \\
&\quad \cup \left[\frac{e_n}{2} + 2^{n-1}\pi, d_n\right] \\
&= I_1 \cup I_2 \cup I_3, \text{ say}
\end{aligned}$$

If $2^{n-1}\xi \in (I_1 \cup I_3)$, then $2^j(2^{n-1}\xi) \in F_n$ only when $j = 0$ (from the table) So, $H(\xi) = |\hat{\psi}_n(2^{n-1}\xi)|^2 = 1$ Also if $2^{n-1}\xi \in I_2$, then $2^j(2^{n-1}\xi) \in F_n$ only when $j = 0$ or 1 So $H(\xi) = |\hat{\psi}_n(2^{n-1}\xi)|^2 + |\hat{\psi}_n(2^n\xi)|^2 = (\frac{1}{\sqrt{2}})^2 + (-\frac{1}{\sqrt{2}})^2 = 1$ Hence, $H(\xi) = 1$ for a e $\xi > 0$

For $\xi < 0$, it suffices to show that $H(\xi) = 1$ on $[-b_n, -a_n]$ In fact, from the table it is clear that if $\xi \in [-b_n, -a_n]$, then $2^j\xi \in F_n$ only when $j = 0$ So, $H(\xi) = |\hat{\psi}_n(\xi)|^2 = 1$ for a e $\xi \in [-b_n, -a_n]$ Thus, we have proved that $H(\xi) = 1$ for a e $\xi \in \mathbb{R}$

Proof of (3)

We now have to prove that

$$t_q(\xi) = \sum_{j \geq 0} \hat{\psi}_n(2^j\xi)\overline{\hat{\psi}_n\left(2^j(\xi + 2q\pi)\right)} = 0 \quad \text{for a e } \xi \in \mathbb{R} \text{ and for all } q \in 2\mathbb{Z} + 1$$

Since $t_q(\xi) = \overline{t_{-q}(\xi + 2q\pi)}$, it is enough to show that $t_q = 0$ a e , if q is a negative odd integer Suppose $q \neq -1$, and odd We have $2^jq \neq 0, \pm 2^{n-1} \pm 2^{n-2}$ Therefore, if $2^j\xi F_n$, then from the table it is clear that $2^j\xi + 2\ 2^jq\pi \notin F_n$, which shows that each term of the sum $t_q(\xi)$ is 0 Hence, $t_q = 0$ a e

It remains to prove that $t_{-1}(\xi) = 0$ for a e. $\xi \in \mathbb{R}$

$$t_{-1}(\xi) = \sum_{j \geq 0} \hat{\psi}_n(2^j\xi)\overline{\hat{\psi}_n(2^j\xi - 2\ 2^j\pi)}$$

From the table (see the 8th and 6th row), we observe that both $2^j\xi$ and $2^j\xi - 2\ 2^j\pi$ belong to F_n only in the following two cases

(i) $2^j\xi \in [a_n + 2^n\pi, e_n + 2^n\pi]$ and $j = n-1$

(ii) $2^j\xi \in [\frac{a_n}{2} + 2^{n-1}\pi, \frac{e_n}{2} + 2^{n-1}\pi]$ and $j = n-2$

But both are equivalent to saying that $2^{n-1}\xi \in [a_n + 2^n\pi, e_n + 2^n\pi]$ So we get $t_{-1}(\xi) = 0$, if $2^{n-1}\xi \notin [a_n + 2^n\pi, e_n + 2^n\pi]$

Now if $2^{n-1}\xi \in [a_n + 2^n\pi, e_n + 2^n\pi]$, then

$$\begin{aligned} t_{-1}(\xi) &= \hat{\psi}_n(2^{n-2}\xi)\overline{\hat{\psi}_n(2^{n-2}\xi - 2^{n-1}\pi)} + \hat{\psi}_n(2^{n-1}\xi)\overline{\hat{\psi}_n(2^{n-1}\xi - 2^n\pi)} \\ &= \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\ \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

We have proved that $t_q(\xi) = 0$ for a e $\xi \in \mathbb{R}$ and for all $q \in 2\mathbb{Z}+1$ Therefore, ψ_n is a wavelet

Our claim now is that $\psi_n \in \mathcal{M}_{n-2}$ For this, we have to show that

(a) $F_n \cap (F_n + 2^k q\pi) = \emptyset$ for all $q \in 2\mathbb{Z}+1$ and $k = 1, 2, \quad n-2$, and

(b) there exists a subset E of F_n such that $E + 2^{n-1}q\pi \subset F_n$ for some $q \in 2\mathbb{Z}+1$

As $2^k q\pi = 2\ 2^{k-1}q\pi$, by referring to the table we observe that if $\xi \in F_n$, then $\xi + 2\ 2^{k-1}q\pi \notin F_n$ (since q is odd and $k-1 \leq n-3$) So (a) is proved For (b) consider $E = [\frac{a_n}{2}, \frac{e_n}{2}]$ and $q = 1$ Then, $E + 2^{n-1}q\pi = [\frac{a_n}{2}, \frac{e_n}{2}] + 2^{n-1}\pi \subset F_n$

Since $n \geq 3$, we have proved that the equivalence classes $M_n, n \geq 1$ are non-empty

3.3.2 A Family of Wavelets belonging to the class $\mathcal{M}_0$

For $n \geq 3$, let $a_n, b_n, c_n, d_n,$ and e_n be as above In Theorem 2 1 of Chapter 2, we define b on $[e_n, b_n]$ as follows

$$b(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \xi \in [e_n, \pi] \\ 0 & \text{if } \xi \in [\pi, b_n] \end{cases}$$

Then we extend b to the whole of S_n by using $(i)-(iv)$ of Theorem 2 1 Observe that $[e_n, b_n] \cap supp\ b \cap \left(\frac{1}{2^{n-1}} supp\ b\right) = [e_n, \pi]$ We define θ on $[e_n, \pi]$ as

$$\theta(\xi) = \begin{cases} \pi & \text{if } \xi \in [e_n, \pi] \\ 0 & \text{otherwise} \end{cases}$$

Clearly, θ satisfies the functional equation in (v) of Theorem 2 1 This choice of b and θ will give us the wavelet w_n, where

$$\hat{w}_n(\xi) = \begin{cases} 1 & \text{if } \xi \in [-\pi, -a_n] \cup [a_n, e_n] \cup [2^{n-1}\pi, d_n] \\ \frac{1}{\sqrt{2}} & \text{if } \xi \in [-d_n, -2^{n-1}\pi] \cup [-b_n, -\pi] \cup [c_n, 2^{n-1}\pi] \\ -\frac{1}{\sqrt{2}} & \text{if } \xi \in [e_n, \pi] \\ 0 & \text{otherwise} \end{cases}$$

Weber [Web, Theorem 4] also proved that a wavelet $\psi \in \mathcal{L}_1$ only if $supp\ \hat{\psi}$ is not partially self similar with respect to any odd multiple of 2π In view of this result, to show that $w_n \in \mathcal{M}_0 = \mathcal{L}_0 \setminus \mathcal{L}_1$, we only have to find a set H_n in $supp\ \hat{\psi}_n$ such that $H_n + 2q\pi$ is also a subset of $supp\ \hat{\psi}_n$, for some odd integer q The choice $H_n = [-b_n, -\pi]$ and $q = 1$ will do the job, as $[-b_n, -\pi] + 2\pi = [e_n, \pi] \subset supp\ \hat{w}_n$

Chapter 4

Interval Wavelet Sets of $H^2(\mathbb{R})$

4.1 Introduction

The classical Hardy space $H^2(\mathbb{R})$ is the collection of all functions of $L^2(\mathbb{R})$ whose Fourier transform is supported in $\mathbb{R}_+ = (0, \infty)$

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \quad \hat{f}(\xi) = 0 \text{ for a e } \xi < 0\}$$

It is clear that $H^2(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$ As in the case of $L^2(\mathbb{R})$, we can define a wavelet for $H^2(\mathbb{R})$ *A function $\psi \in H^2(\mathbb{R})$ is said to be a wavelet of $H^2(\mathbb{R})$ if the system of functions*

$$\{\psi_{j,k} = 2^{j/2}(2^j \; -k) \quad j, k \in \mathbb{Z}\}$$

forms an orthonormal basis for $H^2(\mathbb{R})$. Such a function ψ will be called an H^2-wavelet An example of an H^2-wavelet is the function whose Fourier transform is the characteristic function of the interval $[2\pi, 4\pi]$ In fact, for a long time this was the only known H^2-wavelet P Auscher [Aus] proved that if $\psi \in H^2(\mathbb{R})$ is such that $|\hat{\psi}|$ is continuous and $|\hat{\psi}(\xi)| = O((1 + |\xi|^{-\alpha-\frac{1}{2}})$ at ∞, for some $\alpha > 0$, then ψ cannot be an H^2-wavelet In particular, there is no band-limited H^2-wavelet such that $|\hat{\psi}|$ is continuous

Similar to the case of $L^2(\mathbb{R})$, all H^2-wavelets are characterized by the following four equations

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = \chi_{\mathbb{R}_+}(\xi) \quad \text{for a e } \xi > 0 \tag{4 1}$$

$$\sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}\,(2^j(\xi+2q\pi))} = 0 \quad \text{for a e } \xi\in\mathbb{R} \text{ and for all } q\in 2\mathbb{Z}+1 \tag{4 2}$$

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+2k\pi)|^2 = 1 \quad \text{for a e } \xi\in\mathbb{R} \tag{4 3}$$

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(\xi+2k\pi)\overline{\hat{\psi}\,(2^j(\xi+2k\pi))} = 0 \quad \text{for a e } \xi\in\mathbb{R} \text{ and for all } j\geq 1 \tag{4 4}$$

An H^2-wavelet ψ will be called an MSF (minimally supported frequency) wavelet if $|\hat{\psi}| = \chi_K$ for some measurable subset K of $\mathbb{R}_+$ The associated set K, which has measure 2π, will be called an H^2-wavelet set It can be proved that (see Theorem 1 2, Chapter 1 for the L^2 case) a function $\psi \in H^2(\mathbb{R})$ such that $|\hat{\psi}| = \chi_K$ for some $K \subset \mathbb{R}_+$ is an H^2-wavelet if and only if (4 1) and (4 3) are satisfied We can rewrite the above result in terms of the wavelet set

Theorem 4.1. *A set $K \subset \mathbb{R}_+$ is an H^2-wavelet set if and only if*

(I) $\{K + 2k\pi \quad k\in\mathbb{Z}\}$ *is an a e partition of* $\mathbb{R}$

(II) $\{2^jK \quad j\in\mathbb{Z}\}$ *is an a e partition of* $\mathbb{R}_+$

An H^2-wavelet set K is said to be an *interval H^2-wavelet set* if it is the union of a finite number of intervals of $\mathbb{R}_+$ In [HKLS] the authors characterized all H^2-wavelet sets which is either an interval or a union of two intervals In fact, the only H^2-wavelet set which is an interval is $[2\pi, 4\pi]$ The H^2-wavelet sets which are union of two disjoint intervals are the following sets

$$K = \left[\frac{2(k+1)}{2^{r+1}-1}\pi, \frac{2k}{2^r-1}\pi\right] \cup \left[\frac{2^{r+1}k}{2^r-1}\pi, \frac{2^{r+2}(k+1)}{2^{r+1}-1}\pi\right],$$

where $r > 0,\ 0 < k < 2(2^r-1),\ r,k\in\mathbb{Z}$ (see Theorem 5 2 of [HKLS])

In this chapter we prove a result on the structure of interval wavelet sets of $H^2(\mathbb{R})$ and then characterize all 3-interval H^2-wavelet sets

4.2 Interval H^2-wavelet sets

We have the following theorem regarding interval H^2-wavelet sets.

Theorem 4.2. *Let $I_s = [a_s, b_s]$, $0 \le s \le m$ and $K = I_0 \cup I_1 \cup \cdots \cup I_m$, with $0 < a_0 < b_0 < a_1 < b_1 < \cdots < a_m < b_m$. Then K is an H^2-wavelet set if and only if*

(1) $I_0 \cup (I_{\sigma(1)} - 2k_1\pi) \cup (I_{\sigma(2)} - 2k_2\pi) \cup \cdots \cup (I_{\sigma(m)} - 2k_m\pi) = [a_o, a_0 + 2\pi]$,

for some permutation $\sigma \in S_m, k_i \in \mathbb{Z}_+, 1 \le i \le m$ such that the right end point of each interval is equal to the left end point of the next interval, and

(2) $I_0 \cup (2^{r_1} I_{\rho(1)}) \cup (2^{r_2} I_{\rho(2)}) \cup \cdots \cup (2^{r_m} I_{\rho(m)}) = [a_0, 2a_0]$,

for some permutation $\rho \in S_m, r_i \in \mathbb{Z}_+, 1 \le i \le m$ such that the right end point of each interval is equal to the left end point of the next interval,

where S_m is the permutation group of $\{1, 2, \cdots, m\}$.

Proof: Suppose K is as in the hypothesis and (1) and (2) hold. Then,

$$\begin{aligned}
\bigcup_{k\in\mathbb{Z}} (K + 2k\pi) &= \bigcup_{k\in\mathbb{Z}} \bigcup_{i=0}^{m} (I_i + 2k\pi) \\
&= \bigcup_{k\in\mathbb{Z}} \Big[\{ I_0 \cup \bigcup_{i=1}^{m} (I_{\sigma(i)} - 2k_i\pi) \} + 2k\pi \Big] \\
&= \bigcup_{k\in\mathbb{Z}} \Big\{ [a_0, a_0 + 2\pi] + 2k\pi \Big\} \\
&= \mathbb{R}
\end{aligned}$$

Also, $\{K + 2k\pi : k \in \mathbb{Z}\}$ is pairwise disjoint. Similarly, $\cup_{j\in\mathbb{Z}} (2^j K) = \mathbb{R}_+$ and $\{2^j K : j \in \mathbb{Z}\}$ is pairwise disjoint. Therefore, K is a wavelet set for $H^2(\mathbb{R})$ by Theorem 4.1.

Conversely, suppose that $K = I_0 \cup \cdots \cup I_m$ is an H^2-wavelet set, where $I_r = [a_r, b_r]$, $0 \le r \le m$. So, (I) and (II) of Theorem 4.1 are satisfied. Note that for each $r, 1 \le r \le m$, there is a unique $k_r \in \mathbb{Z}$ such that $a_r + 2k_r\pi \in [a_0, a_0 + 2\pi]$. But, $a_r + 2k_r\pi \notin [a_0, b_0]$. Because in that case, $I_0 \cap (I_r + 2k_r\pi)$ will contain the interval $[a_r + 2k_r\pi, b_0]$ if $b_0 \le b_r + 2k_r\pi$,

or the interval $[a_r + 2k_r\pi, b_r + 2k_r\pi]$ if $b_r + 2k_r\pi \le b_0$ This will be a contradiction to (I) So, $a_r + 2k_r\pi \in [b_0, a_0 + 2\pi]$ Note that $k_r \ge 0$ as $a_r > b_o$

Claim: $I_r + 2k_r\pi = [a_r, b_r] + 2k_r\pi \subset [b_0, a_0 + 2\pi]$

If not, then $b_0 \le a_r + 2k_r\pi < a_0 + 2\pi < b_r + 2k_r\pi$ Therefore, $a_r + 2(k_r - 1)\pi < a_0 < b_r + 2(k_r - 1)\pi$ But, this implies that $I_0 \cap \big(I_r + 2(k_r - 1)\pi\big)$ contains the interval $[a_0, b_r + 2(k_r - 1)\pi]$ if $b_r + 2(k_r - 1)\pi \le b_0$, or the interval $[a_0, b_0]$ if $b_0 \le b_r + 2(k_r - 1)\pi$ In either case, (I) is violated Hence, the claim is proved Now,

$$\begin{aligned}\bigcup_{k\in\mathbb{Z}}(K + 2k\pi) &= \bigcup_{k\in\mathbb{Z}}\Big[\bigcup_{r=0}^{m}(I_r + 2k\pi)\Big] \\ &= \bigcup_{k\in\mathbb{Z}}\Big[\{(I_0 \cup (I_1 + 2k_1\pi) \cup \quad \cup (I_m + 2k_m\pi)\} + 2k\pi\Big]\end{aligned} \tag{4 5}$$

If there is no $r, 1 \le r \le m$, such that $b_0 = a_r + 2k_r\pi$, then $\{(I_1 + 2k_1\pi) \cup \quad \cup (I_m + 2k_m\pi) + 2k\pi\}$ will be properly contained in $[b_0, a_0 + 2\pi]$ In fact, the set $[a_0, a_0 + 2\pi] \setminus \{(I_0 \cup (I_1 + 2k_1\pi) \cup \quad \cup (I_m + 2k_m\pi)) + 2k\pi\}$ will have positive measure which, in turn, will show that (4 5) cannot be an a e. partition of $\mathbb{R}$ So, there is an index $r, 1 \le r \le m$ such that $b_0 = a_r + 2k_r\pi$ Further such an index is unique For, if there exist indices r and s with $1 \le r, s \le m$ such that $b_0 = a_r + 2k_r\pi = a_s + 2k_s\pi$, then $(I_r + 2k_r\pi) \cap (I_s + 2k_s\pi)$ will contain an interval which will again contradict (I)

Therefore, there is exactly one index i_1 such that $b_0 = a_{i_1} + 2k_1\pi, 1 \le i_1 \le m$ and $k \in \mathbb{Z}$ Now, $I_0 \cup (I_{i_1} + 2k_1\pi) = [a_0, b_1 + 2k_1\pi] \subset [a_0, a_0 + 2\pi]$ In a similar manner we can show that (the role of b_0 is now taken by $b_1 + 2k_1\pi$) there is exactly one index i_2 such that

$$b_{i_1} + 2k_1\pi = a_{i_2} + 2k_2\pi, \quad 1 \le i_2 \le m,\ i_2 \ne i_1, \quad k_2 \in \mathbb{Z}_+$$

Similarly,

$$b_{i_2} + 2k_2\pi = a_{i_3} + 2k_3\pi, \quad 1 \le i_3 \le m,\ i_3 \ne i_1, i_2, \quad k_3 \in \mathbb{Z}_+$$

$$b_{i_{m-1}} + 2k_{m-1}\pi = a_{i_m} + 2k_m\pi, \quad 1 \le i_m \le m,\ i_m \ne i_1, \quad , i_{m-1}, \quad k_m \in \mathbb{Z}_+$$

Now, $b_{i_m} + 2k_m\pi$ has to coincide with $a_0 + 2\pi$ Otherwise $[a_0, a_0 + 2\pi] \setminus \{(I_0 \cup (I_{i_1} + 2k_1\pi) \cup \cup (I_{i_m} + 2k_m\pi)) + 2k\pi\}$ will have positive measure which will contradict (I) So, we have proved (1) of the theorem Similarly, by considering dilations by powers of 2 of the intervals I_r and making use of the partition (II) of $\mathbb{R}_+$, we can prove (2) □

4.3 Construction of 3-interval H^2-wavelet sets

We want to characterize all wavelet sets K of $H^2(\mathbb{R})$ consisting of 3 intervals In view of Theorem 4 2, we have the following result for such a K

Corollary 4.1. *Let $K = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3]$ such that $0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3$ Then, K is an H^2-wavelet set if and only if for some non-negative integers r, s, k, l*

$$\begin{aligned}
&(1)\ \textit{either}\ [T1] && [a_1, b_1] \cup ([a_2, b_2] - 2k\pi) \cup ([a_3, b_3] - 2l\pi) = [a_1, a_1 + 2\pi] \\
& && \textit{with}\ b_1 = a_2 - 2k\pi, b_2 - 2k\pi = a_3 - 2l\pi, b_3 - 2l\pi = a_1 + 2\pi \\
&\qquad\quad \textit{or}\ [T2] && [a_1, b_1] \cup ([a_3, b_3] - 2\pi l) \cup ([a_2, b_2] - 2\pi k) = [a_1, a_1 + 2\pi] \\
& && \textit{with}\ b_1 = a_3 - 2l\pi, b_3 - 2l\pi = a_2 - 2k\pi, b_2 - 2k\pi = a_1 + 2\pi \\
&\textit{and}\ (2)\ \textit{either}\ [D1] && [a_1, b_1] \cup 2^{-r}[a_2, b_2] \cup 2^{-s}[a_3, b_3] = [a_1, 2a_1] \\
& && \textit{with}\ b_1 = 2^{-r}a_2, 2^{-r}b_2 = 2^{-s}a_3, 2^{-s}b_3 = 2a_1 \\
&\qquad\quad \textit{or}\ [D2] && [a_1, b_1] \cup 2^{-s}[a_3, b_3] \cup 2^{-r}[a_2, b_2] = [a_1, 2a_1] \\
& && \textit{with}\ b_1 = 2^{-s}a_3, 2^{-s}b_3 = 2^{-r}a_2, 2^{-r}b_2 = 2a_1
\end{aligned}$$

Thus, in order to characterize all H^2-wavelet sets consisting of three intervals, we have to consider each of the four cases $(T_i, D_j), i, j = 1, 2$ and determine the values of the non-negative integers r, s, k, l such that the corresponding relations (T_i, D_j) hold

THE CASE (T1,D1)

We have $K = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3]$

$$b_1 = a_2 - 2k\pi, \qquad b_2 - 2k\pi = a_3 - 2l\pi, \qquad b_3 - 2l\pi = a_1 + 2\pi \tag{4 6}$$

$$b_1 = 2^{-r}a_2, \qquad 2^{-r}b_2 = 2^{-s}a_3, \qquad 2^{-s}b_3 = 2a_1 \tag{4 7}$$

Since $b_1 < a_2$, we have $k \geq 1$ Now, $b_2 - 2k\pi = a_3 - 2l\pi \Rightarrow a_3 - b_2 = 2(l-k)\pi$ Since $a_3 - b_2 > 0$, we have $l > k$ Thus, $l > k \geq 1$ Similarly, $s > r \geq 1$ Solving (4 6) and (4 7) for a_i, b_i, we get

$$\begin{aligned} a_1 &= \tfrac{2(l+1)}{2^{s+1}-1}\pi, & b_1 &= \tfrac{2k}{2^r-1}\pi, \\ a_2 &= \tfrac{2\,2^r k}{2^r-1}\pi, & b_2 &= \tfrac{2(l-k)}{2^{s-r}-1}\pi, \\ a_3 &= \tfrac{2\,2^{s-r}(l-k)}{2^{s-r}-1}\pi, & b_3 &= \tfrac{2\,2^{s+1}(l+1)}{2^{s+1}-1}\pi \end{aligned}$$

We have to ensure, $0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3$ Clearly, $a_1 > 0, b_1 < a_2$ and $b_2 < a_3$ The conditions $a_1 < b_1, a_2 < b_2$, and $a_3 < b_3$ will imply

(i) $(2^r - 1)l < (2^{s+1} - 1)k - (2^r - 1)$,

(ii) $(2^r - 1)l > (2^s - 1)k$,

(iii) $(2^{r+1} - 1)l < (2^{s+1} - 1)k + 2(2^s - 2^r)$

Also, we can show that (iii) $\Rightarrow$(i) So, we have to consider only (ii) and (iii) Eliminating l from (ii) and (iii), we get $1 \leq k < 2(2^r - 1)$

Thus, to get all H^2-wavelet sets in this case, we proceed as follows

Fix $r \geq 1$, then we have to consider only those k such that $1 \leq k < 2(2^r - 1)$ Consider any such k By fixing an $s \geq r+1$, we determine all integer l satisfying (ii) and (iii) Then, any such combination of r, k, s, l will give rise to a wavelet set

For example, let $r = 1$, then $1 \leq k < 2$ So, $k = 1$ If $s = 2$, then (ii) and (iii) give us $l > 3$ and $3l < 11$ This implies $l \geq 4$ and $l \leq 3$ So, there is no l satisfying (ii) and (iii) If $s = 3$, then (ii) and (iii) imply $l > 7$ and $3l < 27 \Rightarrow l \geq 8, l \leq 8 \Rightarrow l = 8$ So, $r = 1, k = 1, s = 3, l = 8$ give rise to a wavelet set. The corresponding wavelet set is

$$[\pi + \tfrac{1}{5}\pi, 2\pi] \cup [4\pi, 4\pi + \tfrac{2}{3}\pi] \cup [18\pi + \tfrac{2}{3}\pi, 19\pi + \tfrac{1}{5}\pi]$$

If $s = 4$, then we get, $l > 15$ and $3l < 59 \Rightarrow l \geq 16, l \leq 19 \Rightarrow l = 16, 17, 18, 19$ So if we take $r = 1, k = 1, s = 4$, we get wavelet sets for each of the l's, $l = 16, 17, 18, 19$ The wavelet set corresponding to $r = 1, k = 1, s = 4, l = 16$ is

$$[\pi + \tfrac{3}{31}\pi, 2\pi] \cup [4\pi, 4\pi + \tfrac{2}{7}\pi] \cup [34\pi + \tfrac{2}{7}\pi, 35\pi + \tfrac{3}{31}\pi]$$

A short table of (r, k, l, s) is given in Table 1 at the end of this chapter

THE CASE (T2,D2)

We have $K = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3]$ and

$$b_1 = a_3 - 2l\pi, \qquad b_3 - 2l\pi = a_2 - 2k\pi, \qquad b_2 - 2k\pi = a_1 + 2\pi, \tag{4 8}$$

$$b_1 = 2^{-s}a_3, \qquad 2^{-s}b_3 = 2^{-r}a_2, \qquad 2^{-r}b_2 = 2a_1 \tag{4 9}$$

Examining the relations among a_i's and b_i's as in the case (T1,D1), we have $l > k \geq 0$ and $s > r \geq 0$ Solving the equations (4 8) and (4 9) for a_i's and b_i's , we get

$$\begin{aligned} a_1 &= \tfrac{2(k+1)}{2^{r+1}-1}\pi, & b_1 &= \tfrac{2l}{2^s-1}\pi, \\ a_2 &= \tfrac{2(l-k)}{2^{s-r}-1}\pi, & b_2 &= \tfrac{2\,2^{r+1}(k+1)}{2^{r+1}-1}\pi, \\ a_3 &= \tfrac{2\,2^s l}{2^s-1}\pi, & b_3 &= \tfrac{2\,2^{s-r}(l-k)}{2^{s-r}-1}\pi \end{aligned}$$

Again, we have to ensure that $0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3$ Clearly, $0 < a_1$ As $r+1 \leq s$, $a_1 < b_1 \Rightarrow b_2 < a_3$ Also, $a_3 < b_3 \Rightarrow b_1 < a_2$ So, we have to consider the inequalities $a_1 < b_1, a_2 < b_2, a_3 < b_3$ These conditions are equivalent to the following inequalities

(i) $(2^{r+1} - 1)l > (2^s - 1)(k+1)$

(ii) $(2^{r+1} - 1)l < (2^{s+1} - 1)k + 2(2^s - 2^r)$

(iii) $(2^r - 1)l > (2^s - 1)k$

If $k = 0$, then (iii) is trivially satisfied So, we have to consider only (i) and (ii) And, if $k > 0$, then (iii)$\Rightarrow$ (i) So, one has to consider (ii) and (iii) Also, (ii) and (iii) imply $k < 2(2^r - 1)$ If $r = 0$, then $k < 0$, which is not possible. Thus, to get all wavelet sets in this case, we proceed as follows

Fix $r \geq 1$ and consider all k such that $0 \leq k < 2(2^r - 1)$ Take $s > r$ If $k = 0$, determine all l satisfying (i) and (ii), and if $k > 0$, then determine all l which satisfy (ii)

and (iii) For example, $r = 1, k = 0, s = 2$ does not give any wavelet set But if we take $r = 1, k = 0, s = 3$, then we get $l = 3$ The corresponding wavelet set is

$$[\tfrac{2}{3}\pi, \tfrac{6}{7}\pi] \cup [2\pi, 2\pi + \tfrac{2}{3}\pi] \cup [6\pi + \tfrac{6}{7}\pi, 8\pi]$$

Observe that when $k > 0$, the inequalities to be considered are same as in the case (T1,D1) So, the table for (T1,D1) also works for (T2,D2) though we will get different wavelet sets A short table for the case $k = 0$ is given in Table 2

THE CASE (T2,D1)

In this case, we have, $s > r > 0$, $l > k \geq 0$. Solving the equations [T2] and [D1] of Corollary 1 for a_i's and b_i's, we get

$$\begin{aligned}
a_1 &= \tfrac{1}{2^s-1}[(2^s-1)k - (2^r-1)l + 2^s]\pi, \\
b_1 &= \tfrac{1}{2^r-1}[(2^{s+1}-1)k - (2^{r+1}-1)l + 2^{s+1}]\pi, \\
a_2 &= 2[(2^{s+1}-1)k - (2^{r+1}-1)l + 2^{s+1}]\pi, \\
b_2 &= \tfrac{1}{2^s-1}[(2^{s+1}-1)k - (2^r-1)l + 2^{s+1}]\pi, \\
a_3 &= \tfrac{1}{2^r-1}[(2^{s+1}-1)k - (2^r-1)l + 2^{s+1}]\pi, \\
b_3 &= 4[(2^s-1)k - (2^r-1)l + 2^s]\pi
\end{aligned}$$

Clearly, $b_1 < a_2$ and $b_2 < a_3$ The conditions $0 < a_1, a_1 < b_1, a_2 < b_2$ and $a_3 < b_3$ will imply

(i) $(2^s-1)k + 2^s > (2^r-1)l$

(ii) $\left[2^s(2^{s+1}-1) - 2^r(2^s-1)\right]k + (2^{2s+1} - 2^{r+s}) > \left[2^s(2^{r+1}-1) - 2^r(2^r-1)\right]l$

(iii) $(2^s-1)(2^{s+1}-1)k + (2^{2s+1} - 2^{s+1}) < \left[2^s(2^{r+1}-1) - (2^r-1)\right]l$

(iv) $\left[2^{r+1}(2^s-1) - (2^{s+1}-1)\right]k + 2^{s+1}(2^r-1) > (2^r-1)(2^{r+1}-1)l$

One can show (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) So, we have to consider only (ii) and (iii) Eliminating l from (iii) and (iv), we get $k < 2(2^r-1)$ To get all wavelet sets in this case, we apply the

similar procedure adopted in the case (T1,D1) See Table 3 for some acceptable values of (r, k, s, l)

THE CASE (T1,D2)

Here we have, $s > r \geq 0$, $l > k \geq 1$ Solving [T1] and [D2] for a_i's and b_i's, we get

$$
\begin{aligned}
a_1 &= \tfrac{1}{2^r-1}\left[(2^r-1)l-(2^s-1)k+2^r\right]\pi, \\
b_1 &= \tfrac{1}{2^s-1}\left[(2^{r+1}-1)l-(2^{s+1}-1)k+2^{r+1}\right]\pi, \\
a_2 &= \tfrac{1}{2^s-1}\left[(2^{r+1}-1)l-(2^s-1)k+2^{r+1}\right]\pi, \\
b_2 &= 4\left[(2^r-1)l-(2^s-1)k+2^r\right]\pi, \\
a_3 &= 2\left[(2^{r+1}-1)l-(2^{s+1}-1)k+2^{r+1}\right]\pi, \\
b_3 &= \tfrac{1}{2^r-1}\left[(2^{r+1}-1)l-(2^s-1)k+2^{r+1}\right]\pi
\end{aligned}
$$

The conditions $0 < a_1$, $a_1 < b_1$, $a_2 < b_2$ and $a_3 < b_3$ are equivalent to the following inequalities

(i) $(2^r-1)l+2^r > (2^s-1)k$

(ii) $\left[2^s(2^r-1)-2^r(2^{r+1}-1)\right]l+(2^{r+s}-2^{2r+1} < \left[2^s(2^s-1)-2^r(2^{s+1}-1)\right]k$

(iii) $\left[2^{s+1}(2^r-1)-(2^{r+1}-1)\right]l+2^{r+1}(2^s-1) > (2^s-1)(2^{s+1}-1)k$

(iv) $(2^r-1)(2^{r+1}-1)l+2^{r+1}(2^r-1) < \left[2^r(2^{s+1}-1)-(2^s-1)\right]k$

The coefficient of l in (ii) is negative if and only if $s = r+1$, in which case (ii) is trivially satisfied If this coefficient is non-negative, then it can be shown that (iv)$\Rightarrow$ (ii) Also, it can be shown that (iii)$\Rightarrow$ (i) So, we need only (iii) and (iv) As (iii) and (iv) imply that $k < 2(2^r-1)$, the case $r = 0$ is ruled out because $k \geq 1$ Table 4 gives few acceptable values of (r, k, s, l)

Table 1

(T1,D1)

r	k	s	l
1	1	2	-
		3	8
		4	16–19
		5	32–40
		6	64–83
		7	128–168
		8	256-339
		9	512–680
		10	1024–1363
2	1	3	3
		4	6,7
		5	11–16
		6	22–35
		7	43–71
	2	3	5
		4	11,12
		5	21–25
		6	43–53
	5	3,4	–
		5	52
10	1	11	3
		12	5–7
		20	1026–2047
		25	32801–65567
20	1	25	33–63

Table 2

(T2,D2), $k = 0$

r	s	l
1	2	–
	3	3
	4	6–9
	5	11–19
	6	22–41
	7	43–83
2	3	–
	4	3,4
	5	5–7
	6	10–17
	7	19–35
5	6	–
	7	3
	8	5–7
	9	9–15
	10	17–31

Table 3

(T2,D1)

r	k	s	l
1	0	2	–
		3	5
		4	–
		5	21
		6	–
		7	85
		8	–
	1	2	5
		3	–
		4	–
		5	–
2	0	3	2
		4	–
		5	9

Table 4

(T1,D2)

r	k	s	l
1	1	s	$2^s - 2$
2	1	3	–
		4	–
		5	–
		6	20
3	1	4	–
		5	–
		6	8
		7	–
		8	–
		9	72

Chapter 5

Multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$

5.1 Introduction

Consider an orthonormal wavelet of $L^2(\mathbb{R})$ At the j-th resolution level, the orthonormal basis generated by the wavelet has a frequency localization proportional to 2^j For example, if ψ is a band-limited wavelet, then the measure of the support of $(\psi_{jk})^\wedge$ is 2^j times the measure of the support of $\hat{\psi}$, since

$$(\psi_{jk})^\wedge(\xi) = 2^{-\frac{j}{2}}\hat{\psi}(2^{-j}\xi)e^{-i2^{-j}k\xi}, \quad j,k \in \mathbb{Z}$$

So when j is large, the wavelet bases have poor frequency localization Better frequency localization can be achieved by a suitable construction starting from an MRA wavelet basis

Let $\{V_j \quad j \in \mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R})$ with corresponding scaling function φ and wavelet ψ Let W_j be the corresponding wavelet subspaces $W_j = \overline{sp}\{\psi_{jk} \quad k \in \mathbb{Z}\}$ In the construction of a wavelet from an MRA, essentially the space V_1 was split into two orthogonal components V_0 and W_0 Note that V_1 is the closure of the linear span of the functions $\{2^{\frac{1}{2}}\varphi(2\ \ -k) \quad k \in \mathbb{Z}\}$, whereas V_0 and W_0 are respectively the closure of the span of $\{\varphi(\ -k) \quad k\}$ and $\{\psi(\ -k) \quad k\}$ Since $\varphi(2\ \ -k) = \varphi\left(2(\ -\frac{k}{2})\right)$, we see that the

above procedure splits the half integer translates of a function into integer translates of two functions

In fact, the splitting is not confined to V_1 alone we can choose to split W_j, which is the span of $\{\psi(2^j\ \ -k)\ \ k\} = \{\psi\left(2^j(\ -\frac{k}{2^j})\right)\ \ k\}$, to get two functions whose $2^{-(j-1)}k$ translates will span the same space W_j Repeating the splitting procedure j times, we get 2^j functions whose integer translates alone span the space W_j If we apply this to each W_j, then the resulting basis of $L^2(\mathbb{R})$, which will consist of integer translates of a countable number of functions (instead of all dilations and translations of the wavelet ψ), will give us a better frequency localization This basis is called "wavelet packet basis" The concept of wavelet packets was introduced by Coifman, Meyer and Wickerhauser [CMW1, CMW2] For a nice exposition of wavelet packets of $L^2(\mathbb{R})$ with dilation 2, see [HW]

The concept of wavelet packet was subsequently generalized to $\mathbb{R}^d$ by taking tensor products [CM] The non-tensor product version is due to Shen [She] Other notable generalizations are the biorthogonal wavelet packets [CD], non-orthogonal version of wavelet packets [CL], the wavelet frame packets [Che] on $\mathbb{R}$ for dilation 2, and the orthogonal, biorthogonal and frame packets on $\mathbb{R}^d$ by Long and Chen [LC] for the dyadic dilation

In this chapter we generalize these concepts to $\mathbb{R}^d$ for arbitrary dilation matrices and we will not restrict ourselves to one scaling function We will consider the case of those MRAs for which the central space is generated by several scaling functions

Definition 5.1. *A $d \times d$ matrix A is said to be a dilation matrix for $\mathbb{R}^d$ if*

(i) $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$ and

(ii) all eigenvalues λ of A satisfy $|\lambda| > 1$

Property (i) implies that A has integer entries and hence $|\det A|$ is an integer, and (ii) says that the integer is greater than 1 Let $B = A^t$, the transpose of A and $a = |\det A| = |\det B|$

Considering $\mathbb{Z}^d$ as an additive group, we see that $A\mathbb{Z}^d$ is a normal subgroup of $\mathbb{Z}^d$ So we can form the cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ It is a well known fact that the number of distinct cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is equal to $a = |\det A|$ ([GM], [Woj]) A subset of $\mathbb{Z}^d$ which consists of exactly one element from each of the a cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ will be called a **set of digits** for the dilation matrix A Therefore, if K_A is a set of digits for A, then we can write

$$\mathbb{Z}^d = \bigcup_{\mu \in K_A} (A\mathbb{Z}^d + \mu),$$

where $\{A\mathbb{Z}^d + \mu \quad \mu \in K_A\}$ are pairwise disjoint A set of digits for A need not be a set of digits for its transpose For example, for the dilation matrix $M = \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathbb{R}^2$, the set $\{\binom{0}{0}, \binom{1}{0}\}$ is a set of digits for M but not for M^t It is easy to see that if K is a set of digits for A, then so is $K - \mu$, where $\mu \in K$ Therefore, we can assume, without loss of generality, that $0 \in K$

The notion of a multiresolution analysis can be extended to $L^2(\mathbb{R}^d)$ by replacing the dyadic dilation by a dilation matrix and allowing the resolution spaces to be spanned by more than one scaling function

Definition 5.2. *A sequence $\{V_j \,.\, j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ will be called a multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied*

(M1) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$

(M2) $\cup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R}^d)$ *and* $\cap_{j\in\mathbb{Z}} V_j = \{0\}$

(M3) $f \in V_j$ *if and only if* $f(A\,) \in V_{j+1}$

(M4) there exist L functions $\{\varphi_1, \varphi_2, \quad , \varphi_L\}$ in V_0, called the scaling functions, such that $\{\varphi_l(\cdot - k) \quad 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_0

The concept of multiplicity was introduced by Hervé [Her] in his Ph D thesis Since $\{\varphi_l(\ - k) \quad 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_0, it follows from property

$(M3)$ that $\{a^{j/2}\varphi_l(A\cdot -k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_j. Observe that if $f \in L^2(\mathbb{R}^d)$, then

$$(a^{j/2} f(A\cdot -k))^{\wedge}(\xi) = a^{-j/2} e^{-i\langle B^{-j}\xi, k\rangle} \hat{f}(B^{-1}\xi), \quad \xi \in \mathbb{R}^d,\ k \in \mathbb{Z}^d$$

First of all we will prove a lemma, the splitting lemma (see [Dau]), which is essential for the construction of wavelet packets. We need the following facts for the proof of the splitting lemma.

(1) Let $\mathbb{T}^d = [-\pi, \pi]^d$ and $f \in L^1(\mathbb{R}^d)$. Since $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d}(\mathbb{T}^d + 2k\pi)$, we can write

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{T}^d}\Big\{\sum_{k\in\mathbb{Z}^d} f(x + 2k\pi)\Big\}dx \tag{5.1}$$

(2) Let $\{s_k : k \in \mathbb{Z}^d\} \in l^1(\mathbb{Z}^d)$ and K_B be a set of digits for the dilation matrix B. As $\mathbb{Z}^d$ can be decomposed as $\mathbb{Z}^d = \cup_{\mu\in K_B}(B\mathbb{Z}^d + \mu)$, we can write

$$\sum_{k\in\mathbb{Z}^d} s_k = \sum_{\mu\in K_B}\sum_{k\in\mathbb{Z}^d} s_{\mu+Bk} \tag{5.2}$$

(3) Let K_B be a set of digits for B. Define

$$Q_0 = \bigcup_{\mu\in K_B} B^{-1}(\mathbb{T}^d + 2\mu\pi)$$

Since K_B is a set of digits for B, the set Q_0 satisfies $\cup_{k\in\mathbb{Z}^d}(Q_0 + 2k\pi) = \mathbb{R}^d$. This fact, together with $|Q_0| = 1$, implies that $\{Q_0 + 2k\pi : k \in \mathbb{Z}^d\}$ is a pairwise disjoint collection (see Lemma 1 of [GM]). Therefore,

$$\int_{\mathbb{R}^d} f(x)dx = \int_{Q_0}\Big\{\sum_{k\in\mathbb{Z}^d} f(x + 2k\pi)\Big\}dx, \quad \text{for } f \in L^1(\mathbb{R}^d) \tag{5.3}$$

A function f is said to be $2\pi\mathbb{Z}^d$- periodic if $f(x + 2k\pi) = f(x)$ for all $k \in \mathbb{Z}^d$ and for a.e. ξ.

5.2 The splitting lemma

Let $\{\varphi_l : 1 \le l \le L\}$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal system. Let $V = \overline{sp}\{a^{1/2}\varphi_l(A\cdot -k) : l, k\}$. For $1 \le l, j \le L$ and

$0 \leq r \leq a-1$, suppose that there exist sequences $\{h^r_{ljk}\ k \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$ Define

$$f^r_l(x) = \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h^r_{ljk} a^{1/2} \varphi_j(Ax-k) \tag{5 4}$$

Taking Fourier transform of both sides

$$\begin{aligned} \hat{f}^r_l(\xi) &= \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h^r_{ljk} a^{-1/2} e^{-i\langle B^{-1}\xi, k\rangle} \hat{\varphi}_j(B^{-1}\xi) \\ &= \sum_{j=1}^{L} h^r_{lj}(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi), \end{aligned} \tag{5 5}$$

where

$$h^r_{lj}(\xi) = \sum_{k \in \mathbb{Z}^d} a^{-1/2} h^r_{ljk} e^{-i\langle \xi, k\rangle}, \quad 1 \leq l, j \leq L,\ 0 \leq r \leq a-1, \tag{5 6}$$

and h^r_{lj} is $2\pi\mathbb{Z}^d$-periodic and is in $L^2(\mathbb{T}^d)$ Now, for $0 \leq r \leq a-1$, define the $L \times L$ matrices

$$H_r(\xi) = \left(h^r_{lj}(\xi)\right)_{1 \leq l, j \leq L} \tag{5 7}$$

By denoting

$$\Phi(x) = (\varphi_1(x), \quad , \varphi_L(x))^t \tag{5 8}$$

$$\hat{\Phi}(\xi) = (\hat{\varphi}_1(\xi), \ldots, \hat{\varphi}_L(\xi))^t, \tag{5 9}$$

we can write (5 5) as

$$\hat{F}_r(\xi) = H_r(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi), \quad 0 \leq r \leq a-1, \tag{5 10}$$

where $F_r(x) = (f^r_1(x), f^r_2(x), \quad , f^r_L(x))^t$ The following lemma characterizes the orthonormality of $\{\varphi_l(\cdot - k) \cdot\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$

Lemma 5.1. *The system* $\{\varphi_l(- k) \quad 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ *is orthonormal if and only if*

$$\sum_{k \in \mathbb{Z}^d} \hat{\varphi}_j(\xi + 2k\pi)\overline{\hat{\varphi}_l(\xi + 2k\pi)} = \delta_{jl}$$

Proof: The proof is similar to the proof of Proposition 1 1 of Chapter 1 □

Lemma 5.2. (The splitting lemma)

Let $\{\varphi_l \;\; 1 \le l \le L\}$ be functions in $L^2(\mathbb{R}^d)$ such that the system $\{a^{1/2}\varphi_j(A \cdot -k)$ $1 \le j \le L,\;\; k \in \mathbb{Z}^d\}$ is orthonormal Let V be its closed linear span Let K be a set of digits for B Also let f_l^r, H^r be as above Then

$$\{f_l^r(\cdot - k) \;\; 0 \le r \le a-1,\; 1 \le l \le L,\; k \in \mathbb{Z}^d\}$$

is an orthonormal system if and only if

$$\sum_{\mu \in K} H_r(\xi + 2B^{-1}\mu\pi) H_s^*(\xi + 2B^{-1}\mu\pi) = \delta_{rs} I_L \tag{5 11}$$

Moreover, $\{f_l^r(\cdot - k) \;\; r, l, k\}$ is an orthonormal basis of V whenever it is orthonormal

Proof: For $1 \le l, j \le L$, $0 \le r, s \le a-1$ and $p \in \mathbb{Z}^d$, we have

$$
\begin{aligned}
&\langle f_j^r, f_l^s(\cdot - p)\rangle \\
&= \frac{1}{(2\pi)^d}\langle (f_j^r)^\wedge, (f_l^s(\cdot - p))^\wedge\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (f_j^r)^\wedge(\xi)\overline{(f_l^s)^\wedge(\xi)e^{-i\langle p,\xi\rangle}}\,d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\Big(\sum_{m=1}^{L} h_{jm}^r(B^{-1}\xi)\hat\varphi_m(B^{-1}\xi)\Big)\overline{\Big(\sum_{n=1}^{L} h_{ln}^s(B^{-1}\xi)\hat\varphi_n(B^{-1}\xi)\Big)}e^{i\langle p,\xi\rangle}\,d\xi \quad \text{(by(5 5))} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi)\overline{h_{ln}^s(B^{-1}\xi)}\hat\varphi_m(B^{-1}\xi)\overline{\hat\varphi_n(B^{-1}\xi)}e^{i\langle p,\xi\rangle}\,d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\sum_{k\in\mathbb{Z}^d}\sum_{m=1}^{L}\sum_{n=1}^{L}\Big\{h_{jm}^r(B^{-1}(\xi+2k\pi))\overline{h_{ln}^s(B^{-1}(\xi+2k\pi))} \\
&\qquad \hat\varphi_m(B^{-1}(\xi+2k\pi))\overline{\hat\varphi_n(B^{-1}(\xi+2k\pi))}\Big\}e^{i\langle p,\xi+2k\pi\rangle}\,d\xi \quad \text{(by (5 1))} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\sum_{\mu\in K}\sum_{k\in\mathbb{Z}^d}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}(\xi+2(Bk+\mu)\pi))\overline{h_{ln}^s(B^{-1}(\xi+2(Bk+\mu)\pi))} \\
&\qquad \hat\varphi_m(B^{-1}(\xi+2(Bk+\mu)\pi))\overline{\hat\varphi_n(B^{-1}(\xi+2(Bk+\mu)\pi))}e^{i\langle p,\xi\rangle}\,d\xi \quad \text{(by (5 2))} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{ln}^s(B^{-1}\xi+2B^{-1}\mu\pi)} \\
&\qquad \Big\{\sum_{k\in\mathbb{Z}^d}\hat\varphi_m(B^{-1}(\xi+2\mu\pi)+2k\pi)\overline{\hat\varphi_n(B^{-1}(\xi+2\mu\pi)+2k\pi)}\Big\}e^{i\langle p,\xi\rangle}\,d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{ln}^s(B^{-1}\xi+2B^{-1}\mu\pi)}\delta_{mn}e^{i\langle p,\xi\rangle}\,d\xi \\
&\qquad\qquad \text{(by Lemma 5 1)} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\Big\{\sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{lm}^s(B^{-1}\xi+2B^{-1}\mu\pi)}\Big\}e^{i\langle p,\xi\rangle}\,d\xi
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\langle f_j^r, f_l^s(\ -p)\rangle &= \delta_{rs}\delta_{jl}\delta_{0p} \\
\Leftrightarrow \sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{lm}^s(B^{-1}\xi+2B^{-1}\mu\pi)} &= \delta_{rs}\delta_{jl} \quad \text{for a e } \xi\in\mathbb{R}^d \\
\Leftrightarrow \sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(\xi+2B^{-1}\mu\pi)\overline{h_{lm}^s(\xi+2B^{-1}\mu\pi)} &= \delta_{rs}\delta_{jl} \quad \text{for a e } \xi\in\mathbb{R}^d \\
\Leftrightarrow \sum_{\mu\in K} H_r(\xi+2B^{-1}\mu\pi)H_s^*(\xi+2B^{-1}\mu\pi) &= \delta_{rs}I_L \quad \text{for a e } \xi\in\mathbb{R}^d
\end{aligned}$$

We have proved the first part of the lemma

Now assume that $\{f_l^r(\ -k)\quad 0\le r\le a-1,\ 1\le l\le L,\ k\in\mathbb{Z}^d\}$ is an orthonormal system We want to show that this is an orthonormal basis of V Let $f\in V$ So there exists $\{c_{jp}\quad p\in\mathbb{Z}^d\}\in l^2(Z^d)$, $1\le j\le L$ such that

$$f(x)=\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(Ax-p)$$

Assume that $f\perp f_l^r(\ -k)$ for all r,l,k

Claim: $f=0$

For all r,l,k such that $0\le r\le a-1$, $1\le l\le L$, $k\in\mathbb{Z}^d$, we have

$$\begin{aligned}
0 &= \langle f_l^r(\ -k), f\rangle = \left\langle f_l^r(\ -k), \sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(A\ \ -p)\right\rangle \\
&= \frac{1}{(2\pi)^d}\left\langle \left(f_l^r(\cdot-k)\right)^\wedge, \left(\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(A\cdot-p)\right)^\wedge\right\rangle \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(f_l^r)^\wedge(\xi)e^{-i\langle k,\xi\rangle}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}a^{-1/2}e^{i\langle B^{-1}\xi,p\rangle}\overline{\hat\varphi_j(B^{-1}\xi)}d\xi \\
&= \frac{a^{-1/2}}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{m=1}^{L} h_{lm}^r(B^{-1}\xi)\hat\varphi_m(B^{-1}\xi)e^{-i\langle k,\xi\rangle}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}e^{i\langle B^{-1}\xi,p\rangle}\overline{\hat\varphi_j(B^{-1}\xi)}d\xi \text{ (by (5 5))} \\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{m=1}^{L} h_{lm}^r(\xi)\hat\varphi_m(\xi)\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\overline{\hat\varphi_j(\xi)}e^{-i\langle k,B\xi\rangle}e^{i\langle p,\xi\rangle}d\xi \quad (\xi\to B\xi) \\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{q\in\mathbb{Z}^d}\sum_{m=1}^{L} h_{lm}^r(\xi+2q\pi)\hat\varphi_m(\xi+2q\pi) \\
&\qquad \sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\overline{\hat\varphi_j(\xi+2q\pi)}e^{-i\langle k,B(\xi+2q\pi)\rangle}e^{i\langle p,\xi+2q\pi\rangle}d\xi \quad \text{(by (5 3))} \\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{m=1}^{L}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi)\overline{c_{jp}}\left\{\sum_{q\in\mathbb{Z}^d}\hat\varphi_m(\xi+2q\pi)\overline{\hat\varphi_j(\xi+2q\pi)}\right\}e^{-i\langle k,B\xi\rangle}e^{i\langle p,\xi\rangle}d\xi
\end{aligned}$$

$$= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi)\overline{c_{mp}}e^{-i\langle k,B\xi\rangle}e^{i\langle p,\xi\rangle}d\xi \quad \text{(by Lemma 5 1)}$$
$$= \frac{a^{1/2}}{(2\pi)^d}\sum_{\mu\in K}\int_{B^{-1}(\mathbb{T}^d+2\mu\pi)}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi)\overline{c_{mp}}e^{-i\langle k,B\xi\rangle}e^{i\langle p,\xi\rangle}d\xi$$
$$= \frac{a^{1/2}}{(2\pi)^d}\sum_{\mu\in K}\int_{B^{-1}\mathbb{T}^d}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi+2B^{-1}\mu\pi)\overline{c_{mp}}e^{-i\left\langle k,B(\xi+2B^{-1}\mu\pi)\right\rangle}e^{i\left\langle p,\xi+2B^{-1}\mu\pi\right\rangle}d\xi$$
$$= \frac{a^{1/2}}{(2\pi)^d}\int_{B^{-1}\mathbb{T}^d}\left\{\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi+2B^{-1}\mu\pi)\overline{c_{mp}}e^{i\left\langle p,\xi+2B^{-1}\mu\pi\right\rangle}\right\}e^{-i\langle k,B\xi\rangle}d\xi$$

Since $\left\{\frac{a^{1/2}}{(2\pi)^d}e^{-i\langle k,B\ \rangle}\ \ k\in\mathbb{Z}^d\right\}$ is an orthonormal basis for $L^2(B^{-1}\mathbb{T}^d)$, the above equations give

$$\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{mp}}e^{i\left\langle \xi+2B^{-1}\mu\pi,p\right\rangle}h_{lm}^r(\xi+2B^{-1}\mu\pi)=0 \text{ a e} \quad \text{for all } r,\ l$$

For $m=1,2,\quad L$, define

$$C_m(\xi)=\sum_{p\in\mathbb{Z}^d}c_{mp}e^{-i\langle\xi,p\rangle}. \tag{5 12}$$

So we have

$$\sum_{\mu\in K}\sum_{m=1}^{L}\overline{C_m(\xi+2B^{-1}\mu\pi)}h_{lm}^r(\xi+2B^{-1}\mu\pi)=0,\quad 0\le r\le a-1,\ \ 1\le l\le L \tag{5 13}$$

Equations (5 11) are equivalent to saying that for $0\le r\le a-1,\ 1\le l\le L$ and for a e $\xi\in\mathbb{R}^d$, the vectors

$$\left(h_{lm}^r(\xi+2B^{-1}\mu\pi)\quad 1\le m\le L,\ \mu\in K\right)$$

are mutually orthogonal and each has norm 1, considered as a vector in the aL-dimensional vector space $\mathbb{C}^{aL}$, so that they form an orthonormal basis for $\mathbb{C}^{aL}$ Equation (5 13) says that the vector

$$\left(C_m(\xi+2B^{-1}\mu\pi)\quad 1\le m\le L,\ \mu\in K\right) \tag{5 14}$$

is orthogonal to each member of the above orthonormal basis of $\mathbb{C}^{aL}$ Hence, the vector in the expression (5 14) is zero In particular, $C_m(\xi)=0$, for all m, $1\le m\le L$ That is, $c_{mp}=0,\ 1\le m\le L,\ p\in\mathbb{Z}^d$ Therefore, $f=0$ This ends the proof □

The splitting lemma can be used to decompose an arbitrary Hilbert space into mutually orthogonal subspaces, as in [CMW2] We will use the following corollary later

Corollary 5.1. *Let $\{E_{lk} \; 1 \leq l \leq L, \; k \in \mathbb{Z}^d\}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Let H^r, $0 \leq r \leq a-1$ be as above and satisfy* (5.11). *Define*

$$F^r_{lk} = \sum_{m=1}^{L} \sum_{p \in \mathbb{Z}^d} h^r_{l,m,p-Ak} E_{mp}, \quad 0 \leq r \leq a-1, \; 1 \leq l \leq L, \; k \in \mathbb{Z}^d$$

Then $\{F^r_{lk} \; 1 \leq l \leq L, \; k \in \mathbb{Z}^d\}$ is an orthonormal basis for its closed linear span $\mathcal{H}^r$ and $\mathcal{H} = \oplus_{r=0}^{a-1} \mathcal{H}^r$

Proof: Let $\varphi_1, \varphi_2, \ldots, \varphi_L$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot - k) \; 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ is an orthonormal system. Let $V = \overline{sp}\{a^{1/2}\varphi_l(A \cdot -k) \; l, \; k\}$. Define a linear operator T from the Hilbert space V to $\mathcal{H}$ by $T(a^{1/2}\varphi_l(A \cdot -k)) = E_{l,k}$. Let f^r_l are as in (5.4). Then, $T(f^r_l(\cdot - k)) = F^r_{l,k}$. Now the corollary follows from the splitting lemma. □

5.3 Construction of multiwavelet packets

Let $\{V_j \; j \in \mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A. Let $\{\varphi_l \, . \; 1 \leq l \leq L\}$ be the scaling functions. Since φ_l, $1 \leq l \leq L$ are in $V_0 \subset V_1$ and $\{a^{1/2}\varphi_j(A\cdot -k), \; 1 \leq j \leq L, \; k \in \mathbb{Z}^d\}$ forms an orthonormal basis of V_1, there exist $\{h_{ljk} \; k \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$, for $1 \leq l, j \leq L$ such that

$$\varphi_l(x) = \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{ljk} a^{1/2} \varphi_j(Ax - k)$$

Taking Fourier transform, we get

$$\begin{aligned} \hat{\varphi}_l(\xi) &= \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{l,j,k} a^{-1/2} e^{-i\langle B^{-1}\xi, k\rangle} \hat{\varphi}_j(B^{-1}\xi) \\ &= \sum_{j=1}^{L} h_{lj}(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi), \end{aligned} \tag{5.15}$$

where $h_{lj}(\xi) = \sum_{k \in \mathbb{Z}^d} a^{-1/2} h_{ljk} e^{-i\langle \xi, k\rangle}$, and h_{lj} is $2\pi\mathbb{Z}^d$-periodic and is in $L^2(\mathbb{T}^d)$. Let $H_0(\xi)$ be the $L \times L$ matrix defined by

$$H_0(\xi) = \Big((h_{lj}(\xi)\Big)_{1 \leq l,j \leq L}.$$

We will call H_0 the **low pass filter matrix** Rewriting (5 15) in the vector notation (5 8) and (5 9), we have

$$\hat{\Phi}(\xi) = H_0(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi) \tag{5 16}$$

Let W_j be the wavelet subspaces, the orthogonal complement of V_j in V_{j+1}

$$W_j = V_{j+1} \ominus V_j$$

Properties (M1) and (M3) of Definition 5.2 now imply that

$$W_j \perp W_{j'}, \quad j \neq j'$$

and

$$f \in W_j \Leftrightarrow f(A^{-j}\,) \in W_0 \tag{5 17}$$

Moreover, by (M2), $L^2(\mathbb{R}^d)$ can be decomposed into orthogonal direct sums as

$$\begin{aligned} L^2(\mathbb{R}^d) &= \bigoplus_{j\in\mathbb{Z}} W_j & (5\ 18)\\ &= V_0 \oplus \Big(\bigoplus_{j\geq 0} W_j\Big) & (5\ 19)\end{aligned}$$

By Lemma 5 1 and equation (5 15), we have (for $1 \leq l, j \leq L$)

$$\begin{aligned}\delta_{jl} &= \sum_{k\in\mathbb{Z}^d} \hat{\varphi}_j(\xi+2k\pi)\overline{\hat{\varphi}_l(\xi+2k\pi)} \\ &= \sum_{k\in\mathbb{Z}^d}\Big\{\sum_{m=1}^{L} h_{jm}(B^{-1}(\xi+2k\pi))\hat{\varphi}_m(B^{-1}(\xi+2k\pi))\Big\} \\ &\qquad \Big\{\sum_{n=1}^{L} \overline{h_{ln}(B^{-1}(\xi+2k\pi))\hat{\varphi}_n(B^{-1}(\xi+2k\pi))}\Big\}\end{aligned}$$

Now, using (5 2), we have

$$\begin{aligned}\delta_{jl} &= \sum_{\mu\in K_B}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{ln}(B^{-1}\xi+2B^{-1}\mu\pi)} \\ &\qquad \sum_{k\in\mathbb{Z}^d}\Big\{\hat{\varphi}_m(B^{-1}(\xi+2\mu\pi)+2k\pi)\overline{\hat{\varphi}_n(B^{-1}(\xi+2\mu\pi)+2k\pi)}\Big\},\end{aligned}$$

where K_B is a set of digits for B Using Lemma 5.1 again, we get

$$\delta_{jl} = \sum_{\mu\in K_B}\sum_{m=1}^{L} h_{jm}(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{lm}(B^{-1}\xi+2B^{-1}\mu\pi)} \tag{5 20}$$

This is equivalent to saying that

$$\sum_{\mu\in K_B} H_0(\xi+2B^{-1}\mu\pi)H_0^*(\xi+2B^{-1}\mu\pi)=I_L \quad \text{for a e } \xi$$

Equation (5 20) is also equivalent to the orthonormality of the vectors

$$\left(h_{lj}(\xi+2B^{-1}\mu\pi) \quad 1\le j\le L,\ \mu\in K_B\right), \quad 1\le l\le L,\ \xi\in\mathbb{T}^d$$

These L orthonormal vectors in the aL-dimensional space $\mathbb{C}^{aL}$ can be completed, by Gram-Schmidt orthonormalization process, to produce an orthonormal basis for $\mathbb{C}^{aL}$ Let us denote the new vectors by

$$\left(h_{lj}^r(\xi+2B^{-1}\mu\pi) \quad 1\le j\le L,\ \mu\in K_B\right), \quad 1\le l\le L,\ 1\le r\le a-1,\ \xi\in\mathbb{T}^d,$$

and extend the functions h_{lj}^r, $1\le r\le a-1$, $1\le l,j\le L$ $2\pi\mathbb{Z}^d$-periodically (see [GLT]) Denoting by $H_r(\xi)$, $1\le r\le a-1$ the $L\times L$ matrix

$$\left(h_{lj}^r(\xi)\right)_{1\le l,j\le L},$$

we have

$$\sum_{\mu\in K_B} H_r(\xi+2B^{-1}\mu\pi)H_s^*(\xi+2B^{-1}\mu\pi)=\delta_{rs}I_L \quad \text{for a e } \xi$$

Now, for $1\le r\le a-1$, $1\le l\le L$, define

$$\hat{f}_l^r(\xi)=\sum_{j=1}^{L} h_{lj}^r(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi) \tag{5 21}$$

Since h_{lj}^r are $2\pi\mathbb{Z}^d$-periodic, there exist $\{h_{ljk}^r \quad k\in\mathbb{Z}^d\}\in l^2(\mathbb{Z}^d)$ such that

$$h_{lj}^r(\xi)=\sum_{k\in\mathbb{Z}^d} a^{-1/2}h_{ljk}^r e^{-i\langle\xi,k\rangle}$$

Now, applying the splitting lemma to V_1, we see that $\{f_l^r(\ -k) \quad 0\le r\le a-1, 1\le l\le L,$ $k\in\mathbb{Z}^d\}$ is an orthonormal basis for V_1 We use the convention $\varphi_l=f_l^0$, $1\le l\le L$ with $h_{lj}=h_{lj}^0$ and $h_{ljk}=h_{ljk}^0$ The decomposition $V_1=V_0\oplus W_0$, and the fact that $\{f_l^0(\cdot-k)$ $1\le l\le L,\ k\in\mathbb{Z}^d\}$ is an orthonormal basis of V_0, imply that

$$\{f_l^r(\cdot-k) \quad 1\le r\le a-1,\ 1\le l\le L,\ k\in\mathbb{Z}^d\}$$

is an orthonormal basis for W_0 By (5 17) and (5 18), we see that

$$\{a^{j/2} f_l^r(A^j \cdot -k) \; 1 \leq r \leq a-1, \; 1 \leq l \leq L, \; j \in \mathbb{Z}, \; k \in \mathbb{Z}^d\}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$ This basis is called the *multiwavelet basis* and the functions $\{f_l^r \; 1 \leq r \leq a-1, \; 1 \leq l \leq L\}$ are the *multiwavelets* associated with the MRA $\{V_j \; j \in \mathbb{Z}\}$ For $0 \leq r \leq a-1$, by denoting $F_r(x) = \left(f_1^r(x), f_2^r(x), \quad , f_L^r(x)\right)^t$ and $\hat{F}_r(\xi) = \left(\hat{f}_1^r(\xi), \hat{f}_2^r(\xi), \quad , \hat{f}_L^r(\xi)\right)^t$, we can write (5 16) and (5 21) as

$$\hat{F}_r(\xi) = H_r(B^{-1}\xi)\hat{\Phi}_r(B^{-1}\xi), \quad 0 \leq r \leq a-1 \tag{5 22}$$

This equation is known as the *scaling relation* satisfied by the scaling functions ($r = 0$) and the multiwavelets ($1 \leq r \leq a-1$)

As we observed, by applying the splitting lemma to the space $V_1 = \overline{sp}\{a^{1/2}\varphi_l(A \cdot -k) \; 1 \leq l \leq L, \; k \in \mathbb{Z}^d\}$, we get the functions f_l^r, $0 \leq r \leq a-1$, $1 \leq l \leq L$ Now, for any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define f_l^n, $1 \leq l \leq L$ recursively as follows Suppose that f_l^r, $r \in \mathbb{N}_0$, $1 \leq l \leq L$ are defined already Then define

$$f_l^{s+ar}(x) = \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{ljk}^s a^{1/2} f_j^r(Ax - k), \quad 0 \leq s \leq a-1, \; 1 \leq l \leq L \tag{5 23}$$

Taking Fourier transform

$$(f_l^{s+ar})^\wedge(\xi) = \sum_{j=1}^{L} h_{lj}^s(B^{-1}\xi)(f_j^r)^\wedge(B^{-1}\xi) \tag{5 24}$$

In vector notation, (5 24) can be written as

$$(F_{s+ar})^\wedge(\xi) = H_s(B^{-1}\xi)\hat{F}_r(B^{-1}\xi) \tag{5 25}$$

Note that (5 23) defines f_l^n for every non-negative integer n and every l such that $1 \leq l \leq L$ Observe that $f_l^0 = \varphi_l, 1 \leq l \leq L$ are the scaling functions and $f_l^r, 1 \leq r \leq a-1, 1 \leq l \leq L$ are the multiwavelets So this definition is consistent with the scaling relation (5 22) satisfied by the scaling functions and the multiwavelets

Definition 5.3. *The functions $\{f_l^n \; n \geq 0, \; 1 \leq l \leq L\}$ as defined above will be called the* **basic multiwavelet packets** *corresponding to the MRA $\{V_j \; j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation A*

The Fourier transforms of the multiwavelet packets

Our aim is to find an expression for the Fourier transform of the basic multiwavelet packets in terms of the Fourier transform of the scaling functions For an integer $n \geq 1$, we consider the unique "**a-adic expansion**" (i e , expansion in the base a)

$$n = \mu_1 + \mu_2 a + \mu_3 a^2 + \cdot \quad + \mu_j a^{j-1}, \tag{5 26}$$

where $0 \leq \mu_i \leq a-1$ for all $i = 1, 2, \ . \ , j$ and $\mu_j \neq 0$

If n can be expressed as in (5 26) then we will say n has a-adic length j We claim that if n has length j and has expansion (5 26), then

$$\hat{F}_n(\xi) = H_{\mu_1}(B^{-1}\xi)H_{\mu_2}(B^{-2}\xi) \quad H_{\mu_j}(B^{-j}\xi)\hat{\Phi}(B^{-j}\xi) \tag{5 27}$$

so that $(f_l^n)^\wedge(\xi)$ is the l-th component of the column vector in the right hand side of (5 27) We will prove the claim by induction

From (5 22) we see that the claim is true for all n of length 1 Assume it for length j Then an integer m of a-adic length $j+1$ is of the form $m = \mu + an$, where $0 \leq \mu \leq a-1$ and n has length j Suppose n has the expansion (5 26) Then from (5 25) and (5 27), we have

$$\begin{aligned}
(F_m)^\wedge(\xi) &= (F_{\mu+an})^\wedge(\xi) \\
&= H_\mu(B^{-1}\xi)F_n(B^{-1}\xi) \\
&= H_\mu(B^{-1}\xi)H_{\mu_1}(B^{-2}\xi) \quad H_{\mu_j}(B^{(-j+1)}\xi)\hat{\Phi}(B^{-(j+1)}\xi)
\end{aligned}$$

Since $m = \mu + an = \mu + \mu_1 a + \mu_2 a^2 + \ \cdot \ + \mu_j a^j$, $\hat{F}_m(\xi)$ has the desired form Hence, the induction is complete

The first theorem regarding the multiwavelet packets is the following

Theorem 5.1. *Let $\{f_l^n \, . \, n \geq 0, \ 1 \leq l \leq L\}$ be the basic multiwavelet packets constructed above Then*

(i) $\{f_l^n(\ - k) \quad a^j \leq n \leq a^{j+1} - 1, \ \ 1 \leq l \leq L, \ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of W_j, $j \geq 0$

(ii) $\{f_l^n(\cdot - k) : 0 \le n \le a^j - 1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ *is an orthonormal basis of* V_j, $j \ge 0$

(iii) $\{f_l^n(\cdot - k) : n \ge 0,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ *is an orthonormal basis of* $L^2(\mathbb{R}^d)$

Proof: Since $\{f_l^n : 1 \le n \le a-1,\ 1 \le l \le L\}$ are the multiwavelets, their $\mathbb{Z}^d$-translates form an orthonormal basis for W_0. So (i) is verified for $j = 0$. Assume for j. We will prove for $j+1$. By assumption, the functions $\{f_l^n(\cdot - k) : a^j \le n \le a^{j+1}-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of W_j. Since $f \in W_j \Leftrightarrow f(A\cdot) \in W_{j+1}$, the system of functions

$$\{a^{1/2} f_l^n(A\cdot - k) : a^j \le n \le a^{j+1} - 1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$$

is an orthonormal basis of W_{j+1}. Let

$$E_n = \overline{sp}\{a^{1/2} f_l^n(A \cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$$

Hence,

$$W_{j+1} = \bigoplus_{n=a^j}^{a^{j+1}-1} E_n \tag{5.28}$$

Applying the splitting lemma to E_n, we get the functions

$$g_l^{n,r}(x) = \sum_{m=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{lmk}^r a^{1/2} f_m^n(Ax - k) \quad (0 \le r \le a-1,\ 1 \le l \le L) \tag{5.29}$$

so that $\{g_l^{n,r}(\cdot - k) : 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of E_n. But by (5.23)

$$g_l^{n,r} = f_l^{r+an}$$

This fact, together with (5.28), shows that

$$\begin{aligned} &\{f_l^{r+an}(\cdot - k) : 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d,\ a^j \le n \le a^{j+1} - 1\} \\ &= \{f_l^n(\cdot - k) : a^{j+1} \le n \le a^{j+2} - 1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\} \end{aligned}$$

is an orthonormal basis for W_{j+1}. So (i) is proved. Item (ii) follows from the observation that $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$ and (iii) follows from the fact that $\overline{\cup V_j} = L^2(\mathbb{R}^d)$. □

5.4 Construction of orthonormal bases from the multiwavelet packets

We now take all dilations by the matrix A and all $\mathbb{Z}^d$-translations of the basic multiwavelet packet functions

Definition 5.4. *Let $\{f_l^n \quad n \geq 0, \quad 1 \leq l \leq L\}$ be the basic multiwavelet packets The collection of functions*

$$\mathcal{P} = \{a^{j/2} f_l^n(A^j \quad -k) \quad n \geq 0, \quad 1 \leq l \leq L, \ j \in \mathbb{Z}, \quad k \in \mathbb{Z}^d\}$$

will be called the **general multiwavelet packets** *associated with the MRA $\{V_j\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L*

Remark 5.1. Obviously the collection $\mathcal{P}$ is overcomplete in $L^2(\mathbb{R}^d)$ For example

(i) The subcollection with $j = 0, \ n \geq 0, \ 1 \leq l \leq L, \ k \in \mathbb{Z}^d$ gives us the basic multiwavelet packet basis constructed in the previous section

(ii) The subcollection with $n = 1, 2, \quad , a-1, \ 1 \leq l \leq L, \ j \in \mathbb{Z}, \ k \in \mathbb{Z}^d$ is the usual multiwavelet basis.

So it will be interesting to find out other subcollections of $\mathcal{P}$ which form orthonormal bases for $L^2(\mathbb{R}^d)$ Our aim is to characterize all such subcollections

For $n \geq 0$ and $j \in \mathbb{Z}$, define the subspaces

$$U_j^n = \overline{sp}\{a^{j/2} f_l^n(A^j \cdot -k) \quad 1 \leq l \leq L, \quad k \in \mathbb{Z}^d\} \tag{5 30}$$

Observe that

$$U_j^0 = V_j \quad \text{and} \quad \bigoplus_{r=1}^{a-1} U_j^r = W_j, \quad j \in \mathbb{Z}$$

Hence, the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{a-1} U_j^r$$

We can generalize this fact to other values of n

Proposition 5.1. *For $n \geq 0$ and $j \in \mathbb{Z}$, we have*

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r} \tag{5 31}$$

Proof: By definition

$$U_{j+1}^n = \overline{sp}\{a^{\frac{j+1}{2}} f_l^n(A^{j+1} \cdot -k) \quad 1 \leq l \leq L, \ k \in \mathbb{Z}^d\}$$

Let

$$E_{l,k}(x) = a^{\frac{j+1}{2}} f_l^n(A^{j+1} \cdot -k), \quad \text{for} \quad 1 \leq l \leq L, \ k \in \mathbb{Z}^d$$

Then $\{E_{l,k} \ \ l, k\}$ is an orthonormal basis of the Hilbert space U_{j+1}^n For $0 \leq r \leq a-1$, let

$$F_{l,k}^r(x) = \sum_{m=1}^{L} \sum_{\beta \in \mathbb{Z}^d} h_{l,m,\beta-Ak}^r E_{m,\beta}(x), \quad 1 \leq l \leq L, \ k \in \mathbb{Z}^d$$

and

$$\mathcal{H}^r = \overline{sp}\{F_{l,k}^r \quad 1 \leq l \leq L, \ k \in \mathbb{Z}^d\}$$

Then, by Corollary 5 1 we have

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} \mathcal{H}^r$$

Now

$$\begin{aligned}
F_{l,k}^r(x) &= \sum_{m=1}^{L} \sum_{\beta \in \mathbb{Z}^d} h_{l,m,\beta-Ak}^r E_{m,\beta}(x) \\
&= \sum_{m=1}^{L} \sum_{\alpha \in \mathbb{Z}^d} h_{l,m,\alpha}^r E_{m,Ak+\alpha}(x) \\
&= \sum_{m=1}^{L} \sum_{\alpha \in \mathbb{Z}^d} h_{l,m,\alpha}^r a^{\frac{j+1}{2}} f_m^n(A^{j+1}x - Ak - \alpha) \\
&= a^{\frac{j}{2}} \sum_{m=1}^{L} \sum_{\alpha \in \mathbb{Z}^d} h_{l,m,\alpha}^r a^{\frac{1}{2}} f_m^n\left(A(A^j x - k) - \alpha\right) \\
&= a^{\frac{j}{2}} f_l^{an+r}(A^j x - k), \quad \text{by (5 23)}
\end{aligned}$$

Therefore,

$$\mathcal{H}^r = U_j^{an+r}$$

and

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r} \quad \square$$

Using Proposition 5 1 we can get various decompositions of the wavelet subspaces W_j, $j \geq 0$, which in turn will give rise to various orthonormal bases of $L^2(\mathbb{R}^d)$

Theorem 5.2. *Let $j \geq 0$ Then, we have*

$$\begin{aligned}
W_j &= \bigoplus_{r=1}^{a-1} U_j^r \\
W_j &= \bigoplus_{r=a}^{a^2-1} U_{j-1}^r \\
&\vdots \\
W_j &= \bigoplus_{r=a^l}^{a^{l+1}-1} U_{j-l}^r, \quad l \leq j \\
W_j &= \bigoplus_{r=a^j}^{a^{j+1}-1} U_0^r,
\end{aligned} \tag{5 32}$$

where U_j^n is defined in (5 30)

Proof: Since $W_j = \bigoplus_{r=1}^{a-1} U_j^r$, we can apply Proposition 5.1 repeatedly to get (5 32) □

Theorem 5 2 can be used to construct many orthonormal bases of $L^2(\mathbb{R}^d)$ We have the following orthogonal decomposition

$$L^2(\mathbb{R}^d) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

For each $j \geq 0$, we can choose any of the decompositions of W_j described in (5 32) For example, if we do not want to decompose any W_j, then we have the usual multiwavelet decomposition. On the other hand, if we prefer the last decomposition in (5 32) for each W_j, then we get the multiwavelet packet decomposition There are other decompositions as well Observe that in (5 32), the lower index of U_j^n's are decreased by 1 in each succesive step If we keep some of these spaces fixed and choose to decompose others by using (5 31), then we get decompositions of W_j which do not appear in (5 32) So there is certain interplay between the indices $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$

Let S be a subset of $\mathbb{N}_0 \times \mathbb{Z}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Our aim is to characterize those S

for which the collection

$$\mathcal{P}_S = \left\{a^{\frac{j}{2}} f_l^n(A^j \cdot -k) : 1 \le l \le L,\ k \in \mathbb{Z}^d,\ (n,j) \in S\right\}$$

will be an orthonormal basis of $L^2(\mathbb{R}^d)$. In other words, we want to find out those subsets S of $\mathbb{N}_0 \times \mathbb{Z}$ for which

$$\bigoplus_{(n,j)\in S} U_j^n = L^2(\mathbb{R}^d) \tag{5.33}$$

By using (5.31) repeatedly, we have

$$\begin{aligned} U_j^n &= \bigoplus_{r=0}^{a-1} U_{j-1}^{an+r} \\ &= \bigoplus_{r=an}^{a(n+1)-1} U_{j-1}^r = \bigoplus_{r=an}^{a(n+1)-1} \left[\bigoplus_{s=0}^{a-1} U_{j-2}^{ar+s}\right] \\ &= \bigoplus_{r=a^2n}^{a^2(n+1)-1} U_{j-2}^r = \cdots = \bigoplus_{r=a^jn}^{a^j(n+1)-1} U_0^r \end{aligned} \tag{5.34}$$

Let $I_{n,j} = \{r \in \mathbb{N}_0 : a^j n \le r \le a^j(n+1) - 1\}$. Hence,

$$U_j^n = \bigoplus_{r\in I_{n,j}} U_0^r$$

That is,

$$\bigoplus_{(n,j)\in S} U_j^n = \bigoplus_{(n,j)\in S} \bigoplus_{r\in I_{(n,j)}} U_0^r$$

But we have already proved in Theorem 5.1 that

$$L^2(\mathbb{R}^d) = \bigoplus_{r\in\mathbb{N}_0} U_0^r$$

Thus, for (5.33) to be true, it is necessary and sufficient that $\{I_{n,j} : (n,j) \in S\}$ is a partition of $\mathbb{N}_0$. We say $\{A_l : l \in I\}$ is a partition of $\mathbb{N}_0$ if $A_l \subset \mathbb{N}_0$, A_l's are pairwise disjoint, and $\cup_{l\in I} A_l = \mathbb{N}_0$. We summarize the above discussion in the following theorem.

Theorem 5.3. *Let $\{f_l^n : n \ge 0,\ 1 \le l \le L\}$ be the basic multiwavelet packets and $S \subset \mathbb{N}_0 \times \mathbb{Z}$. Then the collection of functions*

$$\left\{a^{\frac{j}{2}} f_l^n(A^j \cdot -k) : 1 \le l \le L,\ k \in \mathbb{Z}^d,\ (n,j) \in S\right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$ if and only if $\{I_{n,j} : (n,j) \in S\}$ is a partition of $\mathbb{N}_0$.

5.5 Wavelet frame packets

Let $\mathcal{H}$ be a separable Hilbert space A sequence $\{x_k \; k \in \mathbb{Z}\}$ of $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that for all $x \in \mathcal{H}$

$$C_1 \|x\|^2 \le \sum_{k \in \mathbf{Z}} |\langle x, x_k \rangle|^2 \le C_2 \|x\|^2 \tag{5 35}$$

The largest C_1 and the smallest C_2 for which (5 35) holds are called the frame bounds

Suppose that $\Phi = \{\varphi^1, \varphi^2, \quad, \varphi^N\} \subset L^2(\mathbb{R}^d)$ such that $\{\varphi^l(- k) \; 1 \le l \le N, k \in \mathbb{Z}^d\}$ is a frame for its closed linear span $S(\Phi)$. Let $\psi^1, \psi^2, \quad, \psi^N$ be elements in $S(\Phi)$ So each ψ^j is a linear combination of $\varphi^l(- k)$, $1 \le l \le L$, $k \in \mathbb{Z}^d$ A natural question to ask is When can we say that $\{\psi^l(\cdot - k) \; 1 \le l \le N, \; k \in \mathbb{Z}^d\}$ is also a frame for $S(\Phi)$?

If $\psi^j \in S(\Phi)$, then there exists $\{p_{jlk} \; k \in \mathbb{Z}^d\}$ in $l^2(\mathbb{Z}^d)$ such that

$$\psi^j(x) = \sum_{l=1}^{N} \sum_{k \in \mathbf{Z}^d} p_{jlk} \varphi^l(x - k)$$

In terms of Fourier transform

$$\begin{aligned} \hat{\psi}^j(\xi) &= \sum_{l=1}^{N} \sum_{k \in \mathbf{Z}^d} p_{jlk} e^{-i\langle k, \xi \rangle} \hat{\varphi}^l(\xi) \\ &= \sum_{l=1}^{N} P_{jl}(\xi) \hat{\varphi}^l(\xi) \quad (1 \le j \le N), \end{aligned} \tag{5 36}$$

where $P_{jl}(\xi) = \sum_{k \in \mathbf{Z}^d} p_{jlk} e^{-i\langle k, \xi \rangle}$. Let $P(\xi)$ be the $N \times N$ matrix

$$P(\xi) = \Big(P_{jl}(\xi)\Big)_{1 \le j, l \le N}$$

Let S and T be two positive definite matrices of order N We say $S \le T$ if $\langle x, Sx \rangle \le \langle x, Tx \rangle$ for all $x \in \mathbb{R}^N$ The following lemma is the generalization of Lemma 3 1 in [Che]

Lemma 5.3. *Let φ^l, ψ^l for $1 \le l \le N$, and $P(\xi)$ be as above Suppose that there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that*

$$C_1 I \le P^*(\xi) P(\xi) \le C_2 I \quad \textit{for a.e } \xi \in \mathbb{T}^d \tag{5 37}$$

Then, for all $f \in L^2(\mathbb{R}^d)$, we have

$$C_1 \sum_{l=1}^{N} \sum_{k \in \mathbf{Z}^d} \left|\langle f, \psi^l(- k)\rangle\right|^2 \le \sum_{l=1}^{N} \sum_{k \in \mathbf{Z}^d} \left|\langle f, \varphi^l(- k)\rangle\right|^2 \le C_2 \sum_{l=1}^{N} \sum_{k \in \mathbf{Z}^d} \left|\langle f, \varphi^l(- k)\rangle\right|^2 \tag{5 38}$$

Let A be a dilation matrix, $B = A^*$ and $a = |\det A| = |\det B|$ Let

$$K_A = \{\alpha_0, \alpha_1, \quad , \alpha_{a-1}\} \tag{5 39}$$

and

$$K_B = \{\beta_0, \beta_1, \quad , \beta_{a-1}\} \tag{5 40}$$

be fixed sets of digits for A and B respectively For $0 \le r, s \le a-1$ and $1 \le l, j \le L$, define for a e ξ

$$\mathcal{E}_{lj}^{rs}(\xi) = \delta_{lj} a^{-\frac{1}{2}} e^{-i\langle \xi + 2B^{-1}\beta_s \pi, \alpha_r \rangle} \tag{5 41}$$

Let

$$E^{rs}(\xi) = \left(\mathcal{E}_{lj}^{rs}(\xi)\right)_{1 \le l, j \le L} \tag{5 42}$$

and

$$E(\xi) = \left(E^{rs}(\xi)\right)_{0 \le r, s \le a-1} \tag{5 43}$$

So $E(\xi)$ is block matrix with a blocks in each row and each column, and each block is a square matrix of order L, so that $E(\xi)$ is a square matrix of order aL We have the following lemma which will be useful for the splitting trick for frames

Lemma 5.4. *(i) If $\nu \in K_A$, then* $\sum_{\mu \in K_B} e^{-i2\pi\langle B^{-1}\mu, \nu\rangle} = a\delta_{0\nu}$

(ii) The matrix $E(\xi)$, defined in (5 43), *is unitary*

Proof: Item (i) is the orthogonal relation for the characters of the finite group $\mathbb{Z}^d/B\mathbb{Z}^d$ (see [Rud1]) Observe that the mapping

$$\mu + B\mathbb{Z}^d \mapsto e^{-i2\pi\langle B^{-1}\mu, \nu\rangle}, \quad \nu \in K_A$$

is a character of the (finite) coset group $\mathbb{Z}^d/B\mathbb{Z}^d$ If $\nu = 0$ (i e, if $\nu \in A\mathbb{Z}^d$), then there is nothing to prove Suppose that $\nu \ne 0$, then there exists a $\mu' \in K_B$ such that $e^{-i2\pi\langle B^{-1}\mu', \nu\rangle} \ne 1$ Since K_B is a set of digits for B, so is $K_B - \mu'$ Hence,

$$\sum_{\mu \in K_B} e^{-i2\pi\langle B^{-1}(\mu - \mu'), \nu\rangle} = \sum_{\mu \in K_B} e^{-i2\pi\langle B^{-1}\mu, \nu\rangle} \tag{5 44}$$

Now

$$\begin{aligned}\sum_{\mu\in K_B} e^{-i2\pi\langle B^{-1}\mu,\nu\rangle} &= e^{-i2\pi\langle B^{-1}\mu',\nu\rangle} \sum_{\mu\in K_B} e^{-i2\pi\langle B^{-1}(\mu-\mu'),\nu\rangle} \\ &= e^{-i2\pi\langle B^{-1}\mu',\nu\rangle} \sum_{\mu\in K_B} e^{-i2\pi\langle B^{-1}\mu,\nu\rangle}, \quad \text{by (5 44)}\end{aligned}$$

Therefore,

$$\sum_{\mu\in K_B} e^{-i2\pi\langle B^{-1}\mu,\nu\rangle} = 0 \text{ as } e^{-i2\pi\langle B^{-1}\mu',\nu\rangle} \neq 1$$

To prove (ii), observe that the (r,s)-th block of the matrix $E(\xi)E^*(\xi)$ is

$$\sum_{t=0}^{a-1} E^{rt}(\xi)\left(E^{ts}(\xi)\right)^*$$

The (l,j)-th entry in this block is

$$\begin{aligned}&\sum_{t=0}^{a-1}\sum_{m=1}^{L} \mathcal{E}^{rt}_{lm}(\xi)\left(\mathcal{E}^{ts}_{mj}(\xi)\right)^* \\ =\ &\sum_{t=0}^{a-1}\sum_{m=1}^{L} \delta_{lm}a^{-1/2}e^{-i\langle\xi+2B^{-1}\beta_t\pi,\alpha_r\rangle}\cdot\delta_{jm}a^{-1/2}e^{i\langle\xi+2B^{-1}\beta_t\pi,\alpha_s\rangle} \\ =\ &\sum_{m=1}^{L}\delta_{lm}\delta_{jm}\sum_{t=0}^{a-1}a^{-1}e^{-i\langle\xi+2B^{-1}\beta_t\pi,\alpha_r-\alpha_s\rangle} \\ =\ &\sum_{m=1}^{L}\delta_{lm}\delta_{jm}\delta_{rs}, \quad \text{(by (i) of the lemma)} \\ =\ &\delta_{lj}\delta_{rs}.\end{aligned}$$

This proves that $E(\xi)E^*(\xi) = I$ Similarly, $E(\xi)^*E(\xi) = I$ Therefore, $E(\xi)$ is a unitary matrix □

5.6 Splitting lemma for frame packets

Let $\{\varphi_l \quad 1\le l\le L\}$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\ -k) \quad 1\le l\le L,\ \ k\in\mathbb{Z}^d\}$ is a frame for its closed linear span V For $0\le r\le a-1$ and $1\le l\le L$, suppose that there exist sequences $\{h^r_{ljk}\ .\ \ k\in\mathbb{Z}^d\}\in l^2(\mathbb{Z}^d)$ Define f^r_l as in (5 4) and (5 5) That is,

$$f^r_l(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h^r_{ljk}a^{1/2}\varphi_j(Ax-k) \tag{5 45}$$

Let $H_r(\xi)$ be the matrix defined in (5.7). Let K_A and K_B be respectively fixed sets of digits for A and B as in (5.39) and (5.40). Let $H(\xi)$ be the matrix

$$H(\xi) = \Big(H_r r(\xi + 2B^{-1}\beta_s\pi)\Big)_{0\le r,s\le a-1} \tag{5.46}$$

$H(\xi)$ is a block matrix with a blocks in each row and each column, and each block is of order L so that $H(\xi)$ is a square matrix of order aL. Assume that there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that

$$C_1 I \le H^*(\xi)H(\xi) \le C_2 I \quad \text{for a.e. } \xi \in \mathbb{T}^d \tag{5.47}$$

We can write f_l^r as

$$\begin{aligned} f_l^r(x) &= \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{1/2}\varphi_j(Ax-k) \\ &= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak} a^{1/2}\varphi_j(Ax-\alpha_s-Ak), \text{ by (5.2)} \\ &= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak}\varphi_j^{(s)}(x-k), \end{aligned}$$

where

$$\varphi_j^{(s)}(x) = a^{1/2}\varphi_j(Ax-\alpha_s),\ 0\le s\le a-1 \tag{5.48}$$

Taking Fourier transform, we obtain

$$\begin{aligned} (f_l^r)^\wedge(\xi) &= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak} e^{-i\langle\xi,k\rangle}(\varphi_j^{(s)})^\wedge(\xi) \\ &= \sum_{j=1}^{L}\sum_{s=0}^{a-1} p^{rs}_{lj}(\xi)(\varphi_j^{(s)})^\wedge(\xi), \end{aligned}$$

where $p^{rs}_{lj}(\xi) = \sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak}e^{-i\langle\xi,k\rangle}$. Define

$$P^{rs}(\xi) = \Big(p^{rs}_{lj}(\xi)\Big)_{1\le l,j\le L} \tag{5.49}$$

and

$$P(\xi) = \Big(P^{rs}(\xi)\Big)_{0\le r,s\le a-1} \tag{5.50}$$

Claim.

$$H(\xi) = P(B\xi)E(\xi), \tag{5.51}$$

where $E(\xi)$ is defined in (5 41)-(5 43).

Proof of the claim The (r,s)-th block of the matrix $P(B\xi)E(\xi)$ is the matrix

$$\sum_{t=0}^{a-1} P^{rt}(B\xi)E^{ts}(\xi)$$

The (l,j)-th entry in this block is equal to

$$\begin{aligned}
& \sum_{t=0}^{a-1}\sum_{m=1}^{L} p_{lm}^{rt}(B\xi)\mathcal{E}_{mj}^{ts}(\xi) \\
= & \sum_{t=0}^{a-1}\sum_{m=1}^{L}\sum_{k\in\mathbf{Z}^d} h_{l,m,\alpha_t+Ak}^{r} e^{-i\langle B\xi,k\rangle}\delta_{mj}a^{-1/2}e^{-i\left\langle \xi+2B^{-1}\beta_s\pi,\alpha_t\right\rangle} \\
= & \sum_{t=0}^{a-1}\sum_{k\in\mathbf{Z}^d} h_{l,j,\alpha_t+Ak}^{r} e^{-i\langle B\xi,k\rangle}a^{-1/2}e^{-i\left\langle \xi+2B^{-1}\beta_s\pi,\alpha_t\right\rangle}
\end{aligned}$$

Now, the (l,j)-th entry in the (r,s)-th block of $H(\xi)$ is

$$\begin{aligned}
& h_{lj}^{r}(\xi+2B^{-1}\beta_s\pi) \\
= & a^{-1/2}\sum_{k\in\mathbf{Z}^d} h_{ljk}^{r} e^{-i\left\langle \xi+2B^{-1}\beta_s\pi,k\right\rangle} \\
= & a^{-1/2}\sum_{t=0}^{a-1}\sum_{k\in\mathbf{Z}^d} h_{l,j,\alpha_t+Ak}^{r} e^{-i\left\langle \xi+2B^{-1}\beta_s\pi,\alpha_t+Ak\right\rangle}, \text{ by (5 2)} \\
= & a^{-1/2}\sum_{t=0}^{a-1}\sum_{k\in\mathbf{Z}^d} h_{l,j,\alpha_t+Ak}^{r} e^{-i\left\langle \xi+2B^{-1}\beta_s\pi,\alpha_t\right\rangle}\ e^{-i\langle B\xi,k\rangle}
\end{aligned}$$

So the claim is proved In particular, we have

$$H^*(\xi)H(\xi) = E^*(\xi)P^*(B\xi)P(B\xi)E(\xi) \tag{5 52}$$

Since $E(\xi)$ is unitary by Lemma 5 4, $H^*(\xi)H(\xi)$ and $P^*(B\xi)P(B\xi)$ are similar matrices Let $\lambda(\xi)$ and $\Lambda(\xi)$ respectively be the maximal and minimal eigenvalues of the positive definite matrix $H^*(\xi)H(\xi)$, and let $\lambda = \inf_{\xi}\lambda(\xi)$ and $\Lambda = \sup_{\xi}\Lambda(\xi)$ (It is clear from (5 51) that $\lambda(\xi)$ and $\Lambda(\xi)$ are $2\pi\mathbb{Z}^d$-periodic functions) Suppose $0 < \lambda \le \Lambda < \infty$ Then we have, by (5 47) (in the sense of positive definite matrices),

$$\lambda I \le H^*(\xi)H(\xi) \le \Lambda I \quad \text{for a e. } \xi \in \mathbb{T}^d$$

which is equivalent to

$$\lambda I \le P^*(\xi)P(\xi) \le \Lambda I \quad \text{for a e } \xi \in \mathbb{T}^d$$

Then by Lemma 5 3, for all $g \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} \lambda\sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,\varphi_l^{(s)}(\ -k)\right\rangle\right|^2 &\le \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle g,f_l^s(\ -k)\rangle|^2 \\ &\le \Lambda\sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,\varphi_l^{(s)}(\ -k)\right\rangle\right|^2, \end{aligned} \tag{5 53}$$

where $\varphi_l^{(s)}$ is defined in (5 48) Since

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,a^{1/2}\varphi_l(A\ \ -k)\right\rangle\right|^2=\sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,\varphi_l^{(s)}(\ -k)\right\rangle\right|^2, \tag{5 54}$$

which follows from (5.48), inequality (5 53) can be written as

$$\begin{aligned} \lambda\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,a^{1/2}\varphi_l(A\cdot-k)\right\rangle\right|^2 &\le \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle g,f_l^s(\ -k)\rangle|^2 \\ &\le \Lambda\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,a^{1/2}\varphi_l(A\cdot-k)\right\rangle\right|^2 \end{aligned} \tag{5 55}$$

This is the splitting trick for frames the $A^{-1}\mathbb{Z}^d$-translates of the L dilated functions $\varphi_l(A\ )$, $1\le l\le L$, are 'decomposed' into $\mathbb{Z}^d$-translates of the aL functions f_l^s, $0\le s\le a-1$, $1\le l\le L$

We now apply the splitting trick to the functions $\{f_l^s\cdot\ 1\le l\le L\}$ for each s, $0\le s\le a-1$ We have

$$\begin{aligned} \lambda\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,a^{1/2}f_l^s(A\cdot-k)\right\rangle\right|^2 &\le \sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle g,f_l^{s,r}(\cdot-k)\rangle|^2 \\ &\le \Lambda\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,a^{1/2}f_l^s(A\ \ -k)\right\rangle\right|^2, \end{aligned} \tag{5 56}$$

where $f_l^{s,r}$, $0\le r\le a-1$ are defined as in (5 45) (f_l^s now replaces φ_l)

$$f_l^{s,r}(x)=\sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d}h_{ljk}^s a^{1/2}f_j^r(Ax-k),\ \ 0\le s\le a-1,\ 1\le l\le L \tag{5 57}$$

Summing (5 56) over $0\le s\le a-1$, we have

$$\begin{aligned} \lambda\sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,a^{1/2}f_l^s(A\ \ -k)\right\rangle\right|^2 &\le \sum_{s=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle g,f_l^{s,r}(\ -k)\rangle|^2 \\ &\le \Lambda\sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g,a^{1/2}f_l^s(A\cdot-k)\right\rangle\right|^2 \end{aligned}$$

Using (5 55), we obtain

$$\lambda^2\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{2/2}\varphi_l(A^2\ -k)\rangle\right|^2 \le \sum_{s=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle g, f_l^{s,r}(\ -k)\rangle|^2$$
$$\le \Lambda^2\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{2/2}\varphi_l(A^2\ -k)\rangle\right|^2 \quad (5\ 58)$$

Now as in the case of orthonormal wavelet packets, we can define f_l^n, for each $n \ge 0$ and $1 \le l \le L$ (see (5 23) and (5 27)) In order to ensure that f_l^n are in $L^2(\mathbb{R}^d)$, it is sufficient to assume that all the entries in the matrix $H(\xi)$, defined in (5 46), are bounded functions Comparing (5 57) and (5 23), we see that

$$\{f_l^{s,r}\ \ 0 \le r, s \le a-1\} = \{f_l^{s+ar} \cdot 0 \le r, s \le a-1\} = \{f_l^n\ \ 0 \le n \le a^2-1\}$$

So (5 58) can be written as

$$\lambda^2\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{2/2}\varphi_l(A^2\ -k)\rangle\right|^2 \le \sum_{n=0}^{a^2-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle g, f_l^n(\ -k)\rangle|^2$$
$$\le \Lambda^2\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{2/2}\varphi_l(A^2\ -k)\rangle\right|^2$$

By induction, we get for each $j \ge 1$

$$\lambda^j\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{j/2}\varphi_l(A^j\cdot-k)\rangle\right|^2 \le \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle g, f_l^n(\ -k)\rangle|^2$$
$$\le \Lambda^j\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{j/2}\varphi_l(A^j\ -k)\rangle\right|^2 \quad (5\ 59)$$

We summarize the above discussion in the following theorem.

Note: $\{f_l^n\ \ n \ge 0,\ 1 \le l \le L\}$ will be called the wavelet frame packets

Theorem 5.4. *Let $\{\varphi_l \cdot\ 1 \le l \le L\} \subset L^2(\mathbb{R}^d)$ be such that $\{\varphi_l(\ -k)\ \ 1 \le l \le L, k \in \mathbb{Z}^d\}$ is a frame for its closed linear span V_0, with frame bounds C_1 and C_2 Let $H(\xi), H_r(\xi), \lambda$ and Λ be as above. Assume that all entries of $H_r(\xi + 2B^{-1}\beta_s\pi)$ are bounded measurable functions such that $0 < \lambda \le \Lambda < \infty$. Let $\{f_l^n\ \ n \ge 0,\ 1 \le l \le L\}$ be the wavelet frame packets and let $V_j = \{f\ \ f(A^{-j}\cdot) \in V_0\}$ Then for all $j \ge 0$, the system of functions*

$$\{f_l^n(\ -k)\ \ 0 \le n \le a^j-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$$

is a frame of V_j with frame bounds $\lambda^j C_1$ and $\Lambda^j C_2$

Proof: Since $\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is a frame of V_0 with frame bounds C_1 and C_2, it is clear that for all j

$$\{a^{j/2}\varphi_l(A^j \cdot -k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$$

is a frame of V_j with the same bounds. So from (5.59), we have

$$\lambda^j C_1 \|g\|^2 \le \sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^n(\cdot - k)\rangle|^2 \le \Lambda^j C_2 \|g\|^2, \text{ for all } g \in V_j \tag{5.60}$$

□

In Theorem 5.1 we proved that the basic multiwavelet packets form an orthonormal basis for $L^2(\mathbb{R}^d) = \overline{\cup V_j}$. An analogous result holds for the wavelet frame packets if the matrix $H(\xi)$, defined in (5.46), is unitary.

Before proving this result let us observe how the space $\overline{\cup_{j\ge 0} V_j}$ looks like. Let $V_0 = \overline{sp}\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$, $V_j = \{f : f(A^{-1}\cdot) \in V_0\}$ and $V_j \subset V_{j+1}$. Let $W = \cup V_j$. Then it is easy to check that $f \in W \Rightarrow f(\cdot - A^{-j}k) \in W$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. We claim that elements of the form $A^{-j}k$ are dense in $\mathbb{R}^d$. For $K = \{k_1, k_2, \dots, k_a\}$ a set of digits for A, define the set

$$Q = Q(A, K) = \left\{x \in \mathbb{R}^d : x = \sum_{j\ge 1} A^{-j}\epsilon_j;\ \epsilon_j \in K\right\}$$

In the above representation of x, ϵ_j's need not be distinct. We have

$$\|A^{-j}x\| \le C\alpha^j \|x\|, \quad x \in \mathbb{R}^d,$$

where C is a constant and $0 < \alpha < 1$ (see [Woj, Chapter 5]). For $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $\|x\| = (|x_1|^2 + |x_2|^2 + \dots + |x_d|^2)^{\frac{1}{2}}$. Therefore, the series that defines x is convergent. The set Q satisfies the following properties (see [GM]):

(i) $Q = \cup_{i=1}^{a} A^{-1}(Q + k_i)$

(ii) $\cup_{k\in\mathbb{Z}}(Q + k) = \mathbb{R}^d$

(iii) Q is compact

Let $\epsilon > 0$ and $y \in Q$ We first show that there exists $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ such that $\|y - A^{-j}k\| < \epsilon$ From (i) we have

$$\begin{aligned} Q &= \bigcup_{i=1}^{a} A^{-1}(Q + k_i) \\ &= \bigcup_{i=1}^{a} A^{-1} \left[\bigcup_{m=1}^{a} A^{-1}(Q + k_m) + k_i \right] \\ &= \bigcup_{i=1}^{a} \bigcup_{m=1}^{a} (A^{-2}Q + A^{-2}k_m + A^{-1}k_l) \end{aligned}$$

Hence, for any $j \geq 1$ and any $y \in Q$, there exist $y_j \in Q$ and $l_1, l_2, \ldots, l_j \in K$ such that

$$y = A^{-j}y_j + A^{-j}l_j + A^{-(j-1)}l_{j-1} + \cdots + A^{-1}l_1$$

Therefore,

$$\begin{aligned} \|y - A^{-j}\{l_j + Al_{j-1} + \cdots + A^{j-1}l_1\}\| &= \|A^{-j}y_j\| \\ &\leq C\alpha^j \|y_j\| \\ &\leq C'\alpha^j \quad \text{(as } Q \text{ is compact)} \\ &< \epsilon, \quad \text{choosing } j \text{ suitably} \end{aligned}$$

Now if $y \in \mathbb{R}^d$, then by (ii) $y = y_0 + p$ for some $y_0 \in Q$ and $p \in \mathbb{Z}^d$ For $y_0 \in Q$, there exist $j \geq 0$ and $k \in \mathbb{Z}^d$ such that $\|y_0 - A^{-j}k\| < \epsilon$ That is,

$$\|y_0 + p - A^{-j}(k - A^j p)\| < \epsilon$$

$$\Rightarrow \|y - A^{-j}(k - A^j p)\|)\| < \epsilon$$

So the claim is proved

We have proved that W is invariant under translations by $A^{-j}k$ and these elements are dense in $\mathbb{R}^d$ Therefore, $\overline{W}$ is a closed translation invariant subspace of $L^2(\mathbb{R}^d)$ Hence, $\overline{W} = L^2_E(\mathbb{R}^d)$ for some $E \subset \mathbb{R}^d$ (see [Rud2]), where

$$L^2_E(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \quad supp\ \hat{f} \subset E\}$$

Now let

$$E_0 = \bigcup_{l=1}^{L} \bigcup_{j \geq 0} B^j(supp\ \hat{\varphi}_l)$$

Claim: $E = E_0$ a.e.

To prove the claim we will follow [BDR], Theorem 4.3. Since $\varphi_l(A^j\cdot) \in V_j \subset \overline{W}$, the function $(\varphi_l(A^j\cdot))^\wedge = \frac{1}{a^j}\hat{\varphi}_l(B^{-j}\cdot) \in \widehat{\overline{W}} = \{\hat{f} : f \in \overline{W}\}$. Therefore, $B^j(supp\ \hat{\varphi}_l) = supp\ \left(\frac{1}{a^j}\hat{\varphi}_l(B^j\cdot)\right) \subset E$ for all $j \geq 0$ and $1 \leq l \leq L$, which implies that $E_0 \subset E$. Let $E_1 = E \setminus E_0$. We have

$$f \in V_j \Leftrightarrow \hat{f} = \sum_{l=1}^{L} m_l(B^{-j}\xi)\hat{\varphi}_l(B^{-j}\xi), \tag{5.61}$$

for some $2\pi\mathbb{Z}^d$-periodic functions $m_l \in L^2(\mathbb{T}^d)$. Hence, (5.61) implies that $\hat{f} = 0$ on E_1 for all $f \in V_j$ and hence, for all $f \in \cup V_j = W$. Taking closure, we obtain that $\hat{f} = 0$ on E_1 for all $f \in \overline{W}$. But $\overline{W}$ is the set of all functions whose Fourier transform is supported in E. Since $E_1 \subset E$, we get that $E_1 = \emptyset$ a.e. Therefore, $E = E_0$ a.e. □

Theorem 5.5. *Let $\{\varphi_l(\cdot - k) : 1 \leq l \leq L,\ k \in \mathbb{Z}^d\} \subset L^2(\mathbb{R}^d)$ be a frame for its closed linear span V_0, with frame bounds C_1 and C_2 and let $V_0 \subset V_1$, where $V_j = \{f : f(A^{-j}\cdot) \in V_0\}$. Assume that $H(\xi)$ is unitary for a.e. ξ. Then $\{f_l^n(\cdot - k) : n \geq 0,\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$ is a frame for the space $\overline{\cup_{j\geq 0} V_j}$ with the same frame bounds.*

More generally, let $S = \{(n, j) \in \mathbb{N}_0 \times \mathbb{Z}\}$ be such that $\bigcup_{(n,j)\in S} I_{n,j}$ is a partition of $\mathbb{N}_0$. Then the collection of functions $\{a^{j/2} f_l^n(A^j \cdot - k) : 1 \leq l \leq L,\ (n, j) \in S,\ k \in \mathbb{Z}^d\}$ is a frame for $\overline{\cup_{j\geq 0} V_j}$ with the same bounds C_1 and C_2.

Proof: Since $H(\xi)$ is unitary, $\lambda = \Lambda = 1$ so that the inequalities in (5.59) are equalities, and from (5.60) we have

$$C_1\|g\|^2 \leq \sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^n(\cdot - k)\rangle|^2 \leq C_2\|g\|^2, \quad \text{for all } g \in V_j \tag{5.62}$$

Now let $h \in \overline{\cup_{j\geq 0} V_j}$. Then there exists $h_j \in V_j$ such that $h_j \to h$ as $j \to \infty$. Fix j, then for $j < j'$, we have from (5.62)

$$\sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle h_{j'}, f_l^n(\cdot - k)\rangle|^2 \leq C_2\|h_{j'}\|^2$$

Letting $j' \to \infty$ first and then $j \to \infty$, we have for all $h \in \overline{\cup_{j\geq 0} V_j}$

$$\sum_{n\geq 0}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle h, f_l^n(\cdot - k)\rangle|^2 \leq C_2\|h\|^2$$

To get the reverse inequality we again use (5.62)

$$\begin{aligned} C_1\|h_j\|^2 &\leq \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle h_j, f_l^n(\cdot - k)\rangle|^2 \\ &= \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle h_j - h, f_l^n(\cdot - k)\rangle + \langle h, f_l^n(\cdot - k)\rangle|^2 \end{aligned}$$

Therefore,

$$\begin{aligned} C_1^{1/2}\|h_j\| &\leq \Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle h_j - h, f_l^n(\cdot - k)\rangle|^2\Big)^{\frac{1}{2}} + \Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}|\langle h, f_l^n(\cdot - k)\rangle|^2\Big)^{\frac{1}{2}} \\ &\leq C_2^{1/2}\|h_j - h\| + \Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle h, f_l^n(\cdot - k)\rangle|^2\Big)^{\frac{1}{2}} \end{aligned}$$

Taking $j \to \infty$, we get

$$C_1\|h\|^2 \leq \sum_{n\geq 0}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle h, f_l^n(\cdot - k)\rangle|^2$$

for all $h \in \overline{\cup V_j}$. So the first part is proved.

Now let $U_j^n = \overline{sp}\{a^{j/2} f_l^n(A^j \cdot -k) : 1 \leq l \leq L, \ k \in \mathbb{Z}^d\}$. Then we can prove as in the orthogonal case (see (5.34)) that

$$U_j^n = \bigoplus_{r\in I_{n,j}} U_0^r,$$

where $\bigoplus$ is just a direct sum not necessarily orthogonal, and $I_{n,j} = \{r \in \mathbb{N}_0 : a^j n \leq r \leq a^j(n+1) - 1\}$. Now as $H(\xi)$ is unitary, we have $\lambda = \Lambda = 1$ and hence (5.56) is an equality. Therefore,

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2} f_l^n(A \cdot -k)\right\rangle\right|^2 = \sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^{an+r}(\cdot - k)\right\rangle\right|^2$$

From this we get

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{2/2} f_l^n(A^2 \cdot -k)\right\rangle\right|^2 = \sum_{t=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^{a(an+r)+t}(\cdot - k)\right\rangle\right|^2$$

$$= \sum_{r=a^2 n}^{a^2(n+1)-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^r(\cdot - k)\rangle|^2$$

Similarly,

$$\sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{j/2} f_l^n(A^j \cdot -k)\right\rangle\right|^2 = \sum_{r=a^j n}^{a^j(n+1)-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^r(\cdot - k)\rangle|^2$$
$$= \sum_{r\in I_{n,j}} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^r(\cdot - k)\rangle|^2 \tag{5 63}$$

From the first part of the theorem, we have for all $f \in \overline{\cup V_j}$

$$C_1\|f\|^2 \leq \sum_{n\geq 0} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle f, f_l^n(\cdot - k)\rangle|^2 \leq C_2\|f\|^2$$

But, the set S is such that $\bigcup_{(n,j)\in S} I_{n,j} = \mathbb{N}_0$ Therefore,

$$C_1\|f\|^2 \leq \sum_{(n,j)\in S} \sum_{r\in I_{n,j}} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle f, f_l^r(\cdot - k)\rangle|^2 \leq C_2\|f\|^2$$

Using (5 63), we get

$$C_1\|f\|^2 \leq \sum_{(n,j)\in S} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\left\langle f, a^{j/2} f_l^n(A^j \cdot -k)\right\rangle\right|^2 \leq C_2\|f\|^2$$

for all $f \in \overline{\cup V_j}$. This completes the proof of the theorem □

$$= \sum_{r=a^2 n}^{a^2(n+1)-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^r(\cdot - k)\rangle|^2 .$$

Similarly,

$$\sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{j/2} f_l^n(A^j \cdot - k)\right\rangle\right|^2 = \sum_{r=a^j n}^{a^j(n+1)-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^r(\cdot - k)\rangle|^2$$

$$= \sum_{r\in I_{n,j}} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle g, f_l^r(\cdot - k)\rangle|^2 . \tag{5.63}$$

From the first part of the theorem, we have for all $f \in \overline{\cup V_j}$

$$C_1\|f\|^2 \le \sum_{n\ge 0} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle f, f_l^n(\cdot - k)\rangle|^2 \le C_2\|f\|^2.$$

But, the set S is such that $\bigcup_{(n,j)\in S} I_{n,j} = \mathbb{N}_0$. Therefore,

$$C_1\|f\|^2 \le \sum_{(n,j)\in S} \sum_{r\in I_{n,j}} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} |\langle f, f_l^r(\cdot - k)\rangle|^2 \le C_2\|f\|^2.$$

Using (5.63), we get

$$C_1\|f\|^2 \le \sum_{(n,j)\in S} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\left\langle f, a^{j/2} f_l^n(A^j \cdot - k)\right\rangle\right|^2 \le C_2\|f\|^2$$

for all $f \in \overline{\cup V_j}$. This completes the proof of the theorem. □

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